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INTERNATIONAL SCHOOL OF SPACE SCIENCE

L'Aquila - ITALY

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Web School on: Dynamical Systems and Machine Learning Approaches to Sun-Earth Relations

Dynamical systems approaches and chaos in Sun-Earth relations

Part I. Theoretical framework

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Outline

A. Complex systems: concepts to tackle interdisciplinary problems

B. Multiscale dynamics: decomposition methods

C. Complexity measures: from global to local fractal dimensions

D. Chaotic measures: Lyapunov spectrum and predictability

E. Final remarks

A. Complex systems: concepts to tackle interdisciplinary problems

What is a complex system?

a system composed of **many components** which may **interact with each other**

Which are the features of a complex system?

self-organization

nonlinear dynamics

multiscale dynamics

pattern formation

networks

emergence

[Ott (2002), Strogatz (2014)]

NETWORKS

a **collection** of discrete objects and relationships between (some of) them

SELF-ORGANIZATION

a process where some form of overall order arises from local interactions between parts of an initially disordered system

PATTERN FORMATION

(**statistically**) orderly outcomes of **self-organization**

EMERGENCE

any phenomena which are **difficult or even impossible to predict** from the smaller entities that make up the system

NONLINEAR DYNAMICS

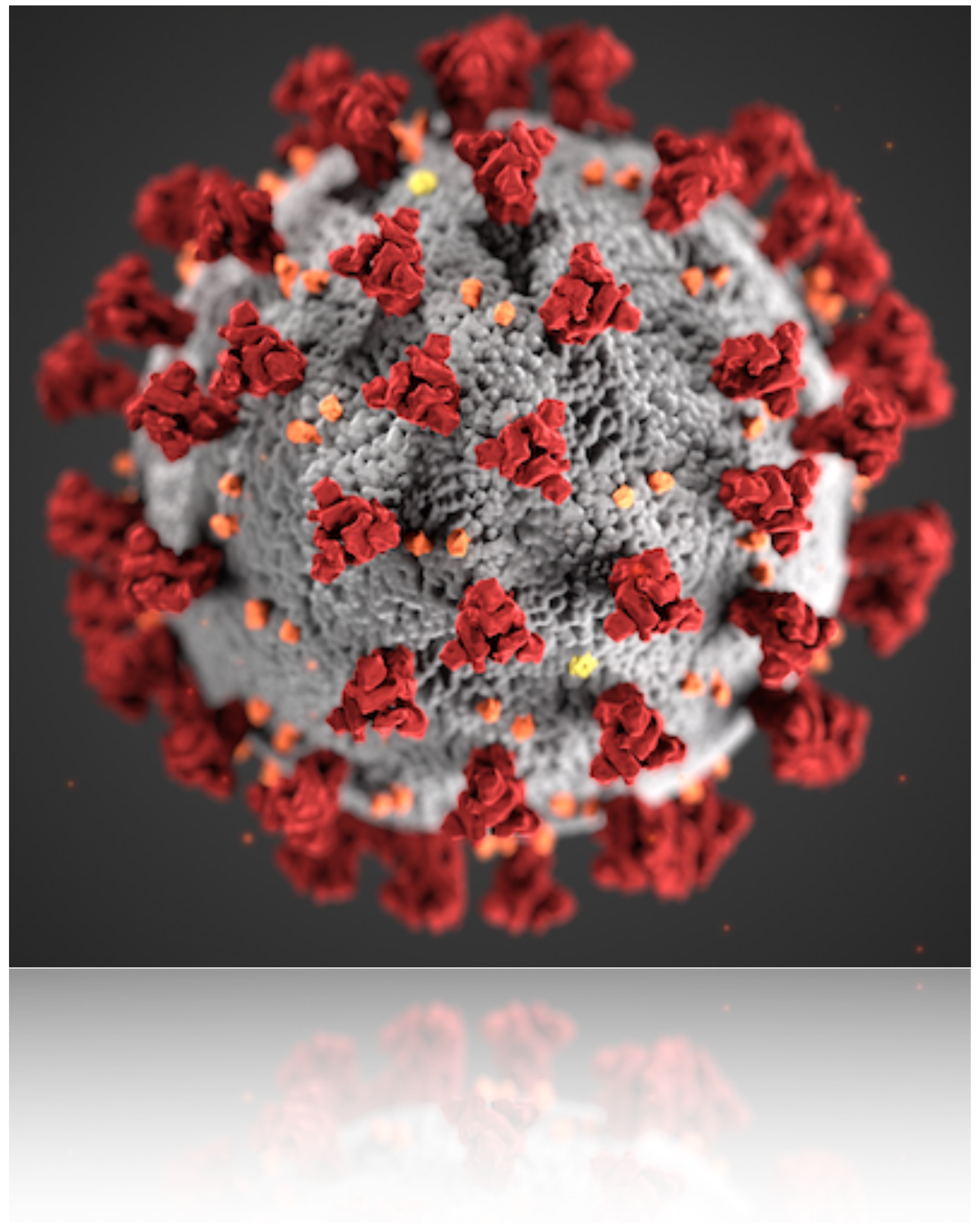
a change in the size of the input does not produce a proportional change in the size of the output

MULTISCALE DYNAMICS

each component of the system evolves on a typical scale that can be **different from the others**

[Ott (2002), Strogatz (2014)]

Complex systems in Nature

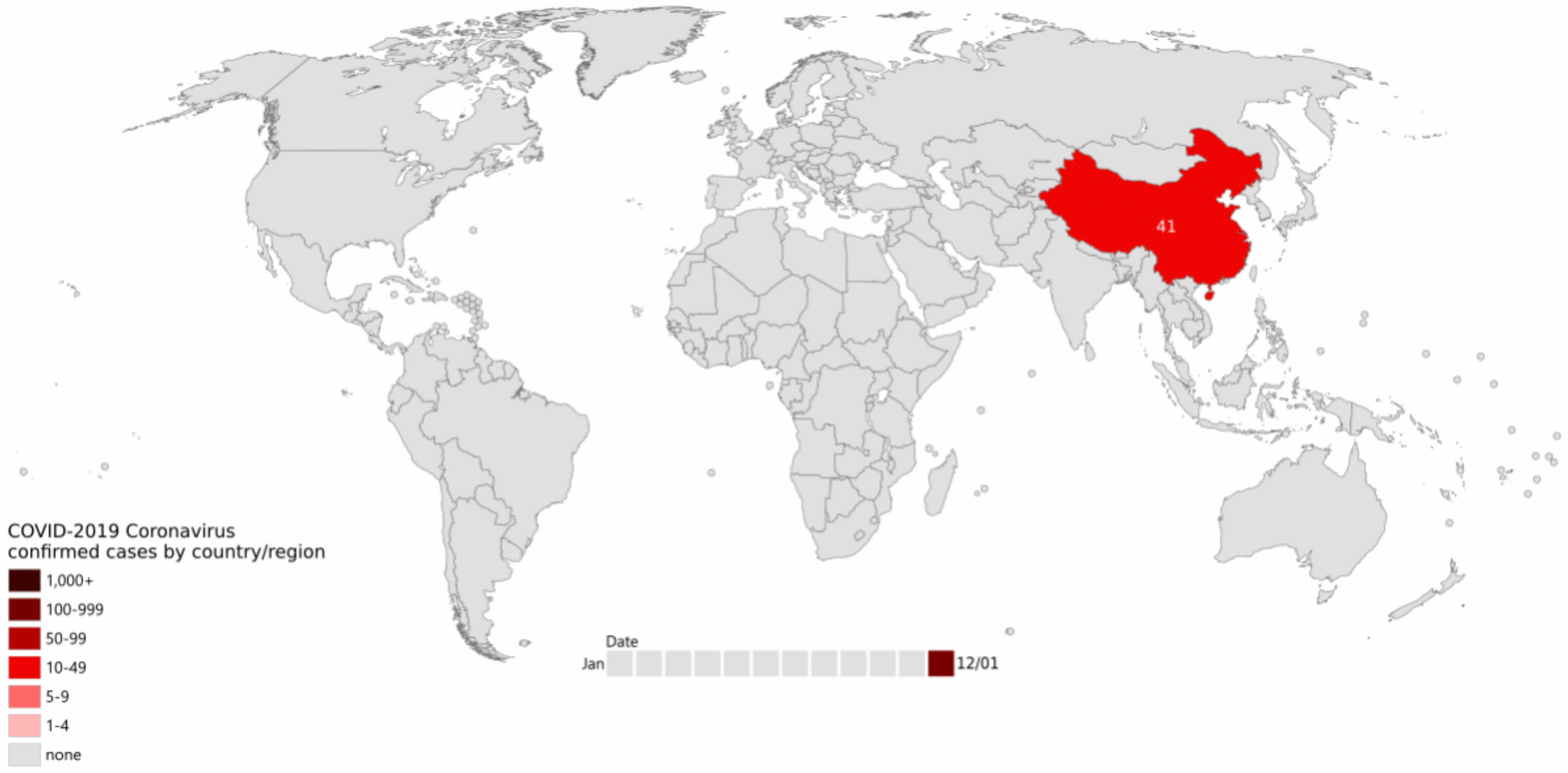


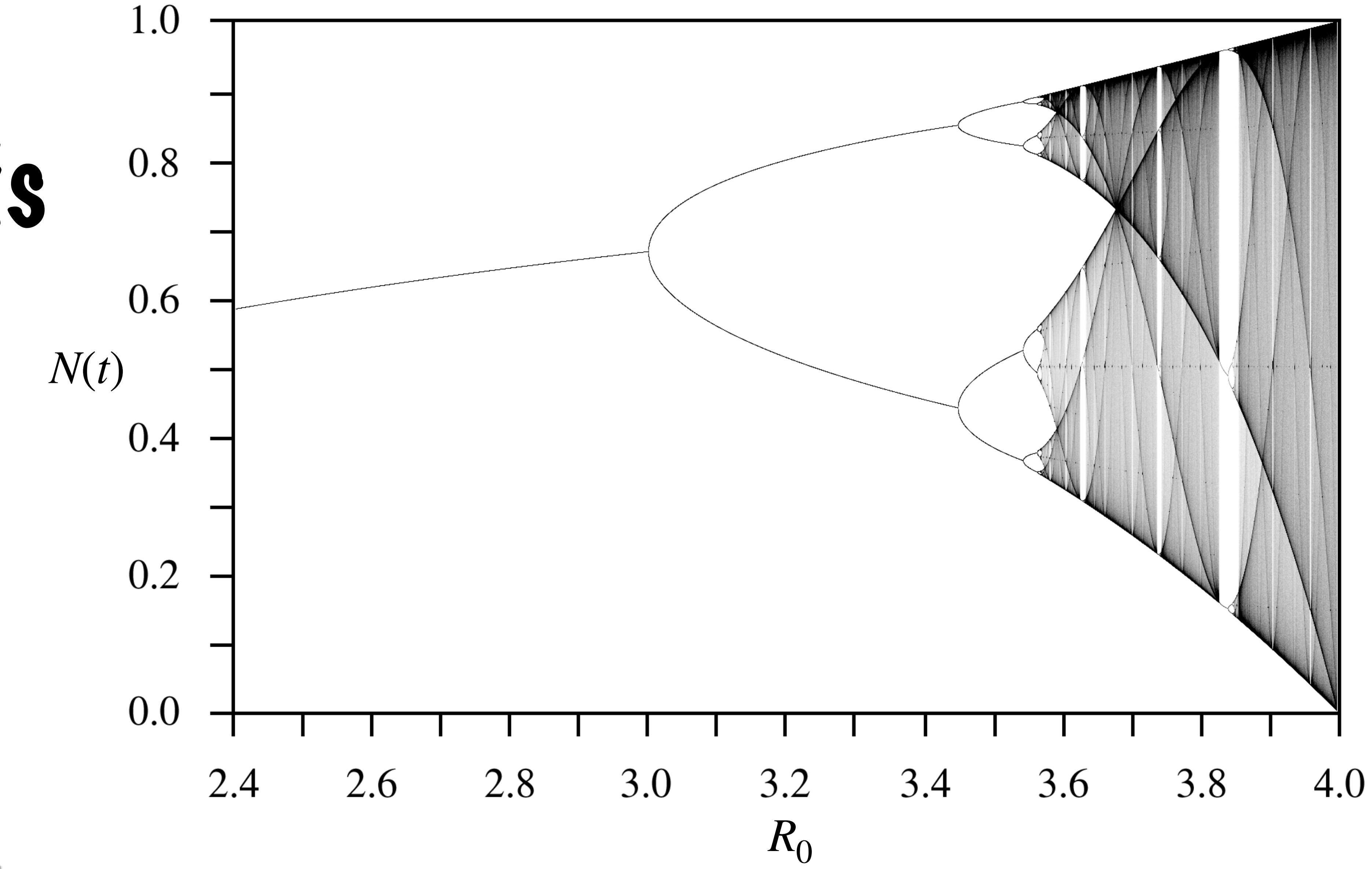
Self-organization:
Take order from disorder
via local interactions

unfortunately here
order means
"infect all people"

COVID-19 pandemic evolution is
a multiscale complex system

$$\frac{dN(t)}{dt} = R_0\, N(t) \left(1 - \frac{N(t)}{N_\infty}\right)$$





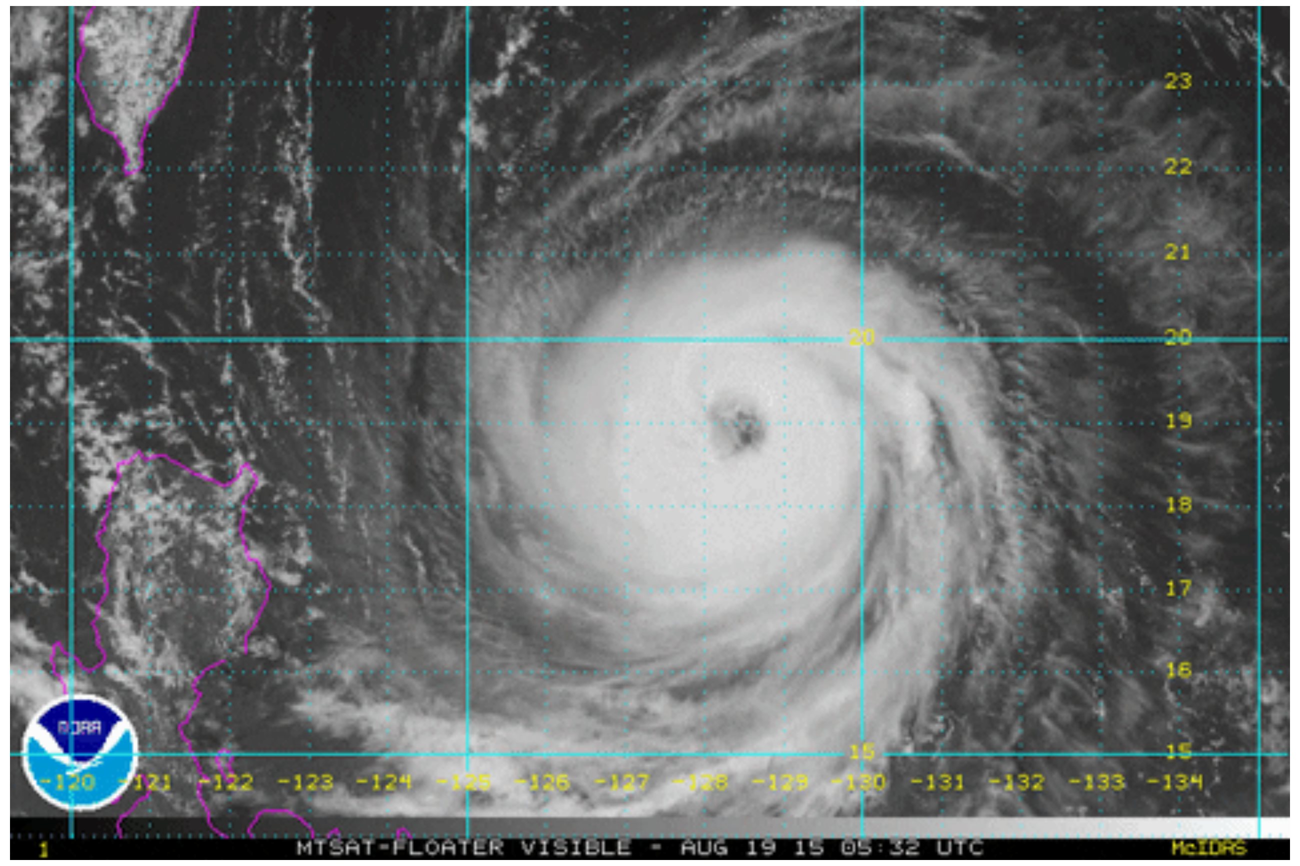
Network:
move one infected person
from a region to another

Nonlinear:
take care of R_0

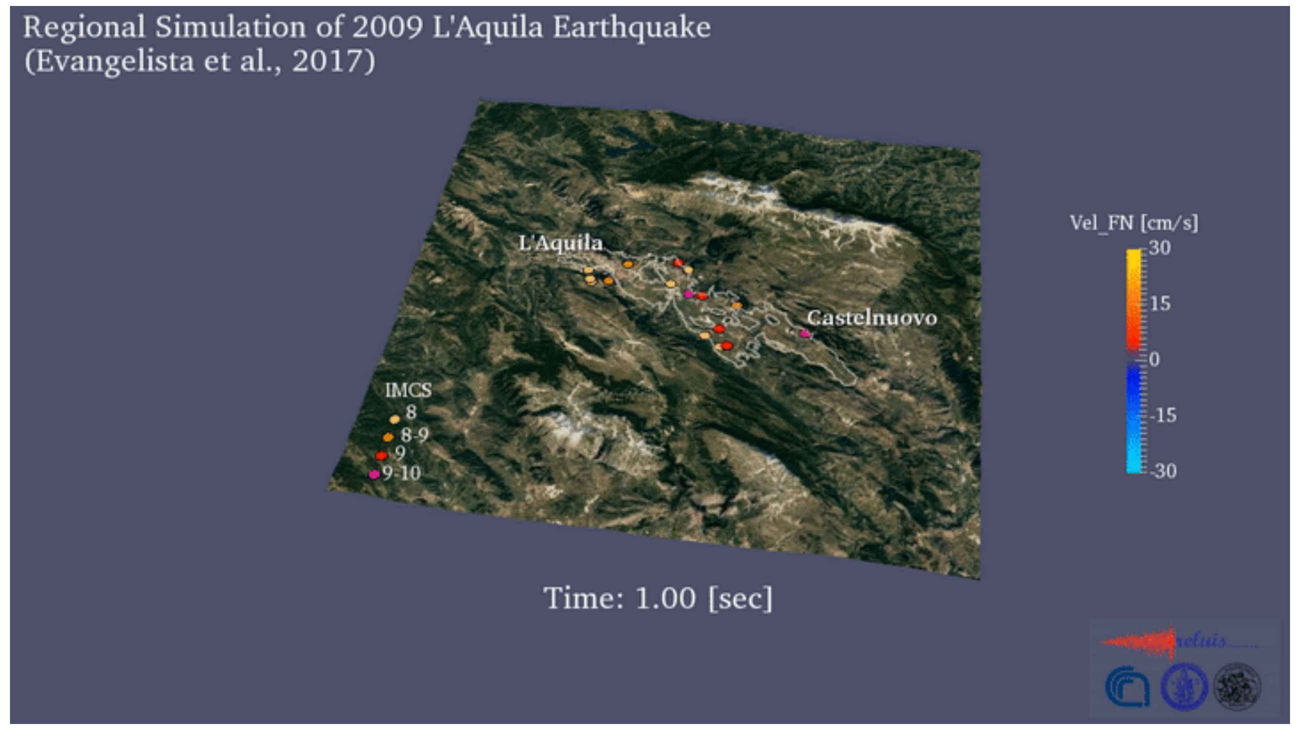
Another one

Aurora borealis

Tropical cyclone



Earthquake



- Many interacting components
- Many forcings
- Wide range of scales
- Many (tele-)connections
- Several nonlinearities

B. Multiscale dynamics: decomposition methods

What is a decomposition method?

a **statistical task** that deconstructs a time series into **several components**, each representing one of the **underlying categories of patterns**

Spectral methods

$$X(t) = \sum_{j,k} c_{jk}\, \phi_{jk}(t)$$

define a suitable spectral basis
for projecting the time series and
extract scale-dependent components

c_{jk} expansion coefficients

$\phi_{jk}(t)$ spectral basis

Adaptive methods

$$X(t) = \sum_{k} \psi_k(t) + T(t)$$

extract a number of components
forming the decomposition basis
with no a priori assumptions

$\psi_k(t)$ decomposition basis

$T(t)$ long-term trend

[Chatfield (2016)]

Spectral methods

I. Wavelet Transform (WT)

mapping from L²(R) → L²(R²) but with superior time-frequency localization as compared with the FT

$$f(t) = C_\psi^{-1} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{da\, db}{a^2} W_f(a,b)\, \psi_{a,b}(t)$$

$$f(t) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle f, \psi_{m,n} \rangle \cdot \psi_{m,n}(t)$$

Properties of WT (both continuous and discrete)

* Define a suitable mother wavelet

$$\psi(t) \in \mathbb{L}^2 : \; C_\psi = \int_0^\infty \frac{\left|\hat{\psi}(\omega)\right|^2}{\left|\omega\right|^2} d\omega < \infty$$

admissibility condition

* Define dilated-translated wavelets

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) \qquad a \in \mathbb{R}^+, \quad b \in \mathbb{R}$$

a: scale parameter

b: translation parameter

* Define wavelet coefficients

$$W_f(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \psi\left(\frac{t-b}{a}\right) f(t)\, dt$$

* Discrete wavelets

$$a = 2^{-m}, \quad b = n\, 2^{-m}, \quad \psi_{m,n}(t) = 2^{\frac{m}{2}} \psi\left(2^m t - n\right)$$

* Satisfied mathematical properties: completeness, orthogonality, Planchener theorem

[Meyer (1992), Chui (1992)]

Spectral methods

I. Wavelet Transform (WT)

How to choose a suitable mother wavelet?

$$\int_{-\infty}^{\infty} \left|\psi(t)\right| dt < \infty \qquad \int_{-\infty}^{\infty} \left|\psi(t)\right|^2 dt < \infty \qquad \int_{-\infty}^{\infty} t^m \psi(t)\, dt = 0$$

* **Haar** Wavelet (1910)

$$\psi(t) = \begin{cases} 1 & 0 \leq t < 1/2, \\ -1 & 1/2 \leq t < 1, \\ 0 & \text{otherwise} \end{cases}$$

- Not continuous
- Not differentiable
- Sudden transitions

* **Morlet** Wavelet (1981)

$$\psi_\sigma(t) = C_\psi\, \pi^{-\frac{1}{4}}\, e^{-\frac{1}{2}t^2} \left(e^{i\sigma t} - e^{-\frac{1}{2}\sigma^2} \right)$$

Complex exponential
+
Gaussian window

* **Mexican hat** Wavelet (1984)

$$\psi(t) = \frac{2}{\sqrt{3\sigma}\, \pi^{\frac{1}{4}}} \left[1 - \left(\frac{t}{\sigma} \right)^2 \right] e^{-\frac{t^2}{2\sigma^2}}$$

Second Hermite function

[Haar (1910), Ricker (1952), Morlet (1981)]

Spectral methods

III. Power Spectral Density (PSD)

how **power** of a signal is distributed over **frequency** -> statistically how **variance** is distributed over **scales**

Let $f(t) : \mathbb{A} \to \mathbb{B}$ be a time series and let $\phi : f(t) \to \phi\left[f(t)\right](\xi), \quad \xi = \{\omega, a, \tau, k, \ldots\},$ some kind of spectral operator (as Fourier or Wavelets) we can define the power spectral density (PSD)

$$S(\xi) \doteq \int_{-\infty}^{\infty} R(\ell)\, e^{-i\,2\pi\xi\,\ell} d\ell = \hat{R}(\xi) \quad \equiv \quad S(\xi) = \lim_{T\to\infty} \frac{1}{T} \left| \phi_T[f(t)](\xi) \right|^2$$

where $R(\ell)$ and $\hat{R}(\xi)$ are the autocorrelation function and its FT, while T is the length of a time window.

Properties of PSD

$S(\xi) \in \mathbb{R}^+$

$S(-\xi) = S(\xi)$

$$\mathrm{Var}(f) = \int_{-\infty}^{\infty} S(\xi)\, d\xi$$

$$S_{f,g}(\xi) = \int_{-\infty}^{\infty} R_{f,g}(\ell) e^{-i\,2\pi\xi\,\ell} d\ell = \hat{R}_{f,g}(\xi)$$

Methods to estimate the PSD

Parametric

- Autoregressive (AR)
- Moving-average (MA)
- Autoregressive moving-average (ARMA)

Non-parametric

- Spectrogram/scalogram
- Welch's method
- Singular spectrum analysis (SSA)

[Chatfield (2016)]

Adaptive methods

I. Empirical Mode Decomposition (EMD)

decomposes signals into **embedded patterns** and a **trend** which are **empirically** determined from the data

$$f(t) = \sum_k \mathscr{C}_k(t) + \mathscr{R}(t)$$

1. Define a zero-mean signal $\qquad f_0(t) = f(t) - \langle f(t) \rangle$

2. Find its local maxima and minima $\quad t_e \in [0,T] \quad s.t. \quad \frac{df_0(t)}{dt}\Big|_{t=t_e} = 0$

3. Define upper and lower envelopes via cubic spline $\qquad \mathscr{U}(t), \mathscr{L}(t)$

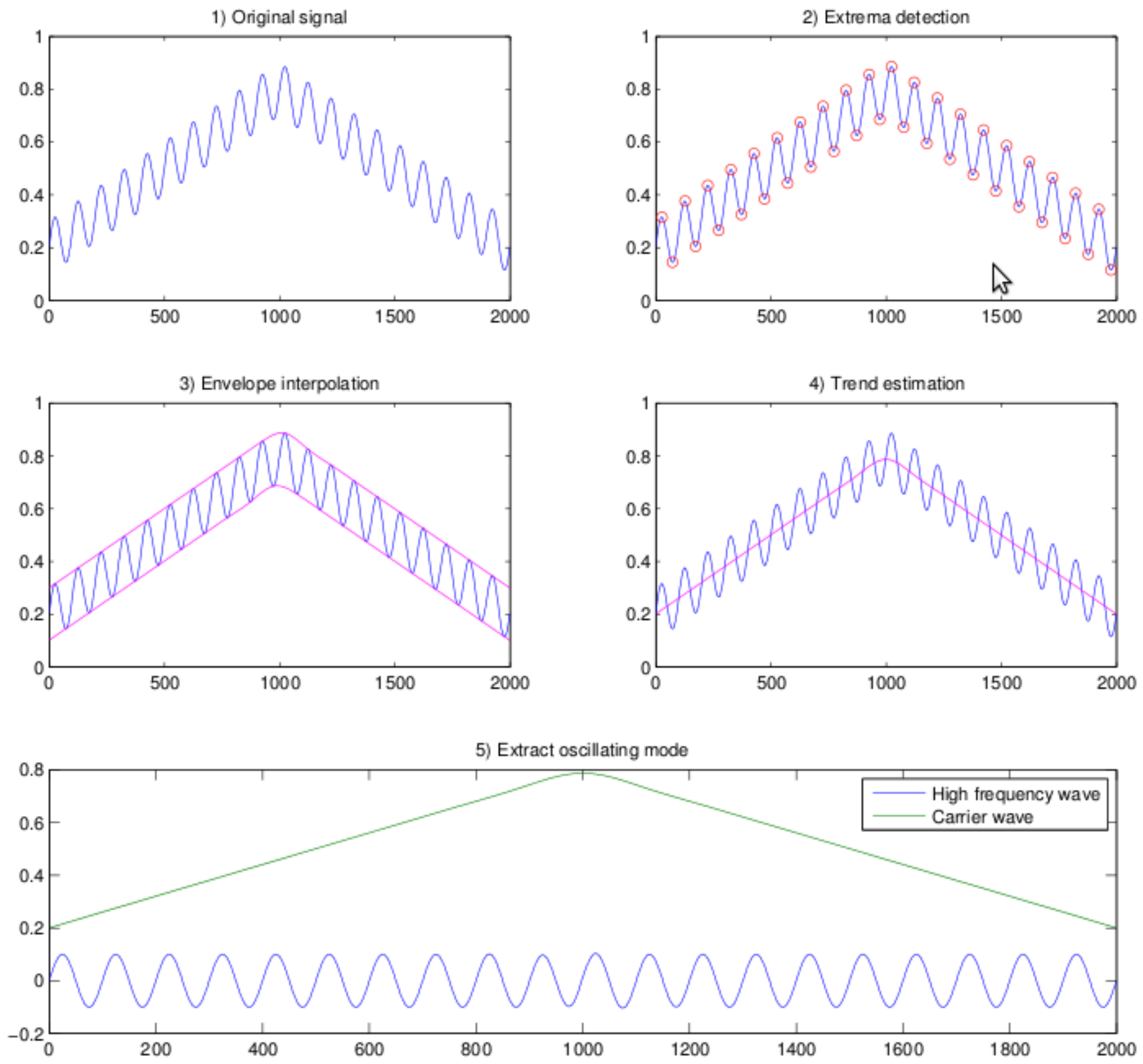
4. Evaluate the mean envelope $\qquad \mathscr{M}(t) = \langle \mathscr{U}(t), \mathscr{L}(t) \rangle$

5. Evaluate the "detail" $\qquad \mathscr{D}(t) = f_0(t) - \mathscr{M}(t)$

6. Is $\mathscr{D}(t)$ an Intrinsic Mode Function (IMF)?
   Does it has (i) the same number of extrema and zero crossing and (ii) an average envelope with zero mean?

   6.1 **YES** -> store $\mathscr{C}_1(t) = \mathscr{D}(t)$ and repeat steps 1.-5. on $f_1(t) = f(t) - \mathscr{D}(t)$

   6.2 **NO** -> repeat steps 1.-5. on $f_1(t) = f_0(t) - \mathscr{D}(t)$ until $\mathscr{D}(t)$ is an IMF



[Huang+ (1998), Huang and Wu (2008)]

Adaptive methods

II. Hilbert Spectral Analysis (HSA)

allows to explore **amplitude-time-frequency** properties of Intrinsic Mode Functions derived via the EMD

$$f(t) = \sum_k \mathscr{C}_k(t) + \mathscr{R}(t) \quad \rightarrow \quad \tilde{\mathscr{C}}_k(t) \doteq \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{\infty} \frac{\mathscr{C}_k(t)}{t - t'} dt' \quad \rightarrow \quad \mathscr{C}_k(t) = \mathscr{A}_k(t)\, e^{i\,\varphi_k(t)}$$

* The combination of both EMD and HT is usually called **Hilbert-Huang Transform (HHT)**
* Define an Hilbert-based spectrogram known as **Hilbert-Huang Spectrum**

$$\mathscr{E}(t,\omega) = \int_{-\infty}^{\infty} \rho(\mathscr{A},\omega)\, \mathscr{A}^2\, d\mathscr{A}$$

* Define an Hilbert-based power spectrum known as **Hilbert-Huang marginal spectrum**

$$\mathscr{P}(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} \mathscr{E}(t',\omega)\, dt'$$

* Define an Hilbert-based PSD known as **Hilbert-Huang PSD**

$$\mathscr{S}(\omega) = \frac{d}{d\omega} \mathscr{P}(\omega)$$

[Huang+ (1998), Huang and Wu (2008)]

C. Complexity measures:
from global to local fractal dimensions

What is a fractal?

self-similar subset of an Euclidean space whose fractal dimension does not exceeds its topological dimension

zooming and zooming in
to uncover finer details but no new detail appears

a measure of roughness
how the details change with the scale

the smallest number n such that
each point of the space belongs to,
at most, n sets in the cover

Clouds are not spheres, mountains are not cones, and lightning does not travel in a straight line (B. Mandelbrot)

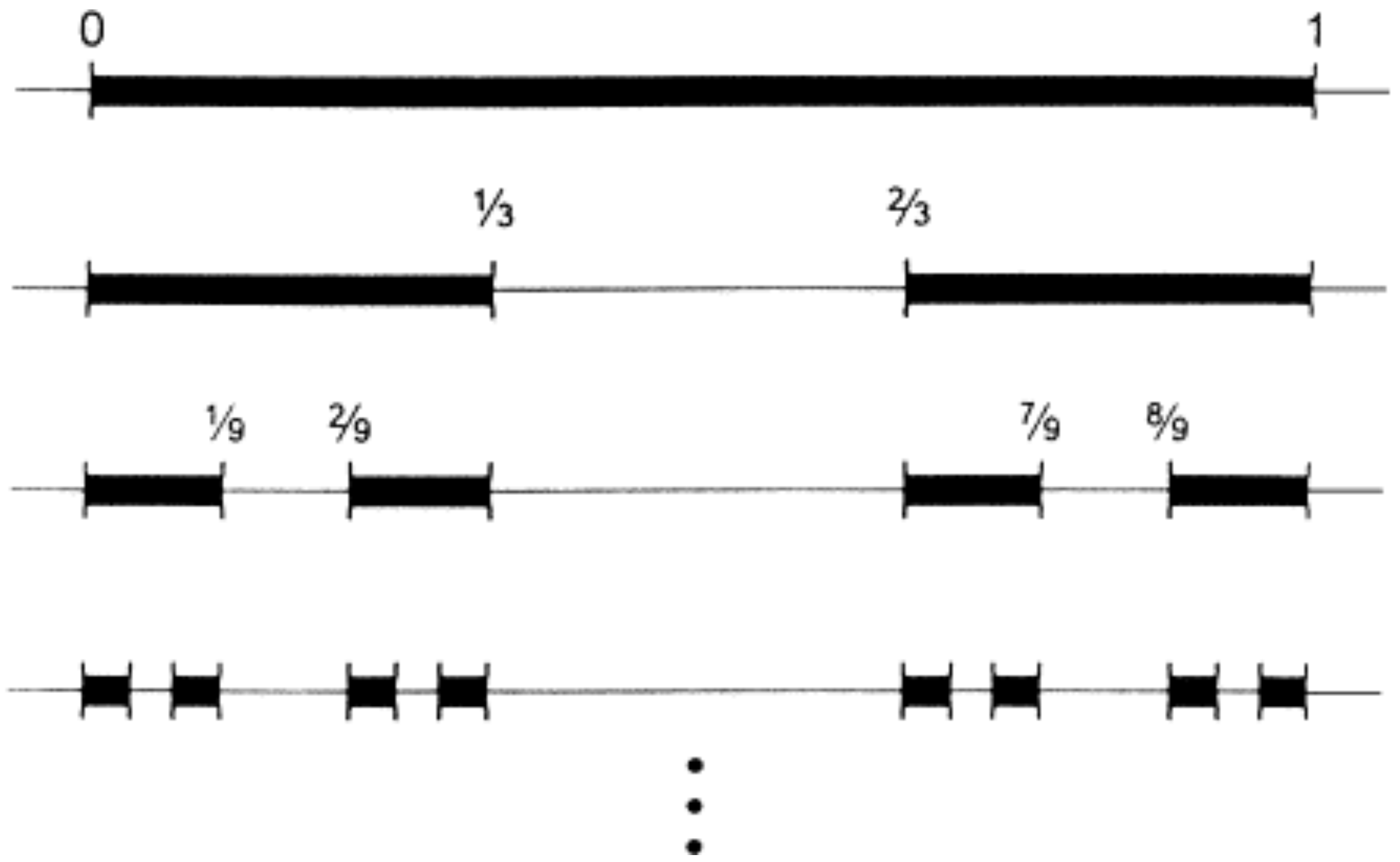
[Mandelbrot (1982), Falconer (2003)]

Building a fractal...

Amazingly, fractals are extremely simple to make!

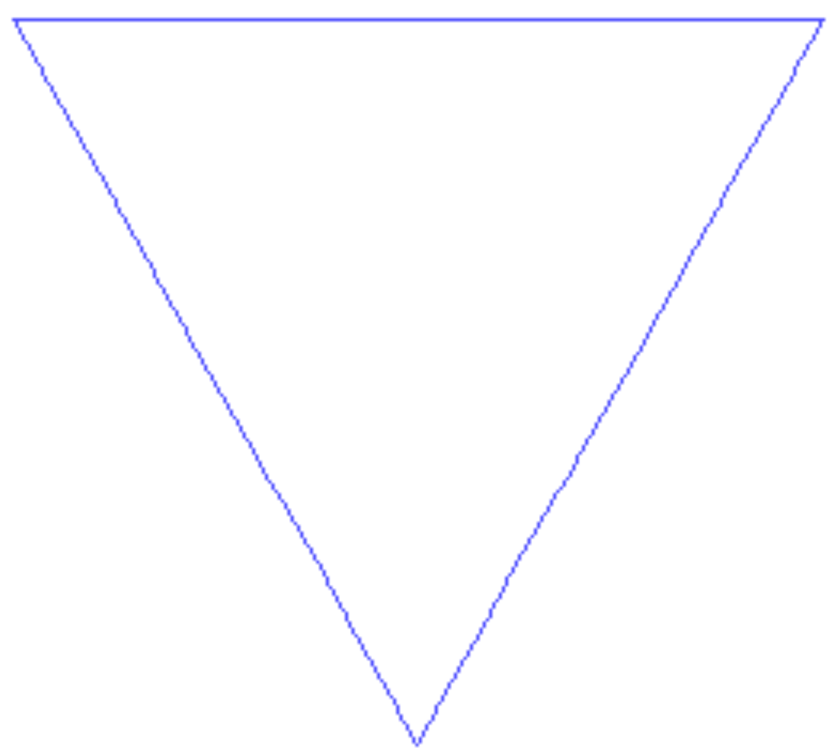
Cantor set

iterative1y deleting the open middle third
from a set of line segments



Koch snowflake

1. start with an equilateral triangle;
2. divide the line segment into three segments of equal length;
3. draw an equilateral triangle that has the middle segment from step 1;
4. remove the line segment that is the base of the triangle from step 2



How many copies (N) of the new object are needed to cover the full-scale one if it is scaled by a factor k?

$$d_F = \frac{\log N}{\log k} = \begin{cases} \frac{\log 2}{\log 3} \approx 0.63\ldots & \text{Cantor} \\ \frac{\log 4}{\log 3} \approx 1.26\ldots & \text{Koch} \end{cases}$$

[Cantor (1883), Koch (1904), Hausdorff (1918)]

...or finding it in nature!

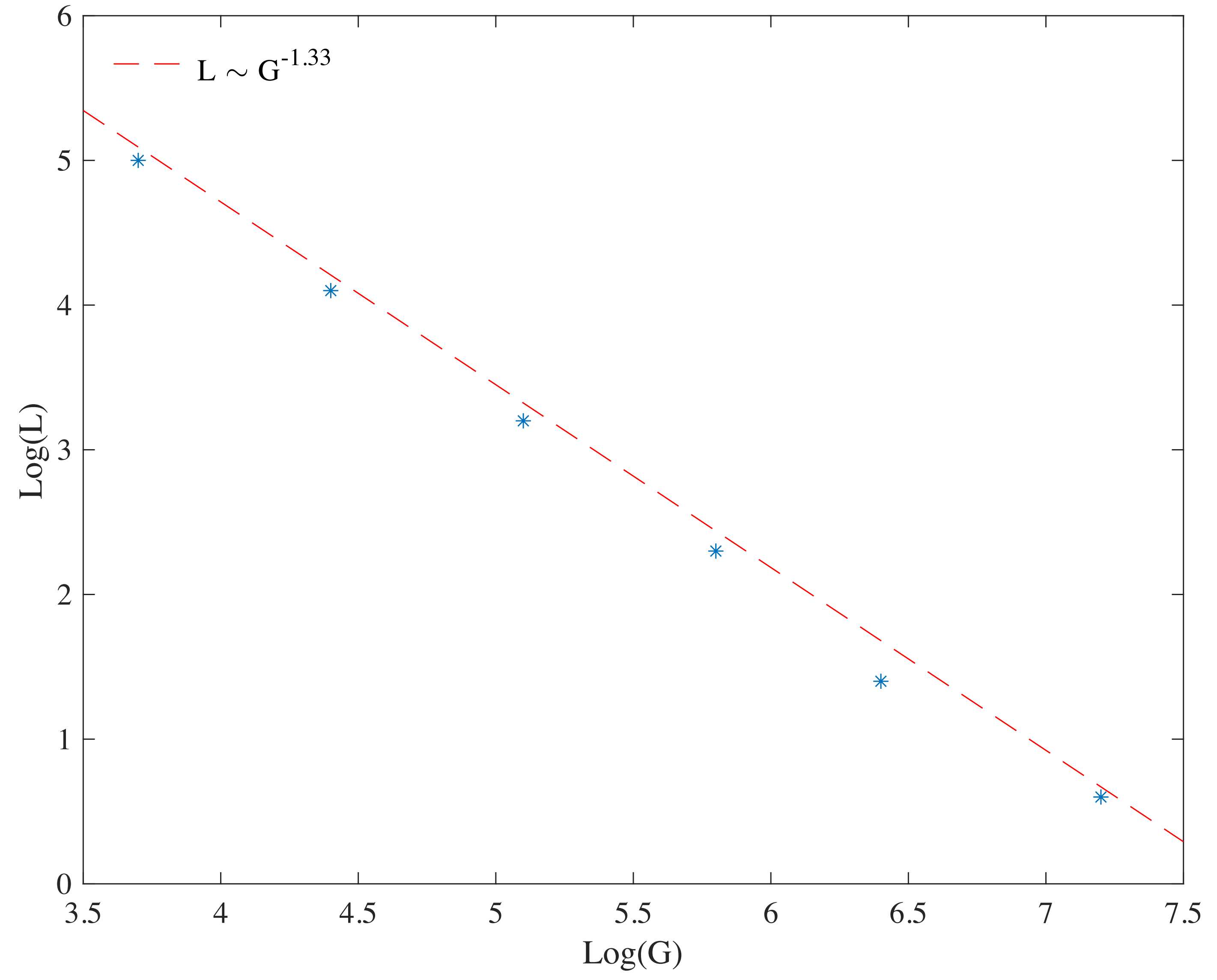
How Long Is the Coast of Britain?

$$L(G) = F\,G^{-D}$$

* It was an older problem dating back in 1960s

* Solutions: L.F. Richardson (1961), B. Mandelbrot (1966)

1. Define a scale parameter G
2. Evaluate how many segments of length G are needed to cover the full coastline
3. Make a plot of $L(G)$ vs. G
4. Find the scaling exponent D

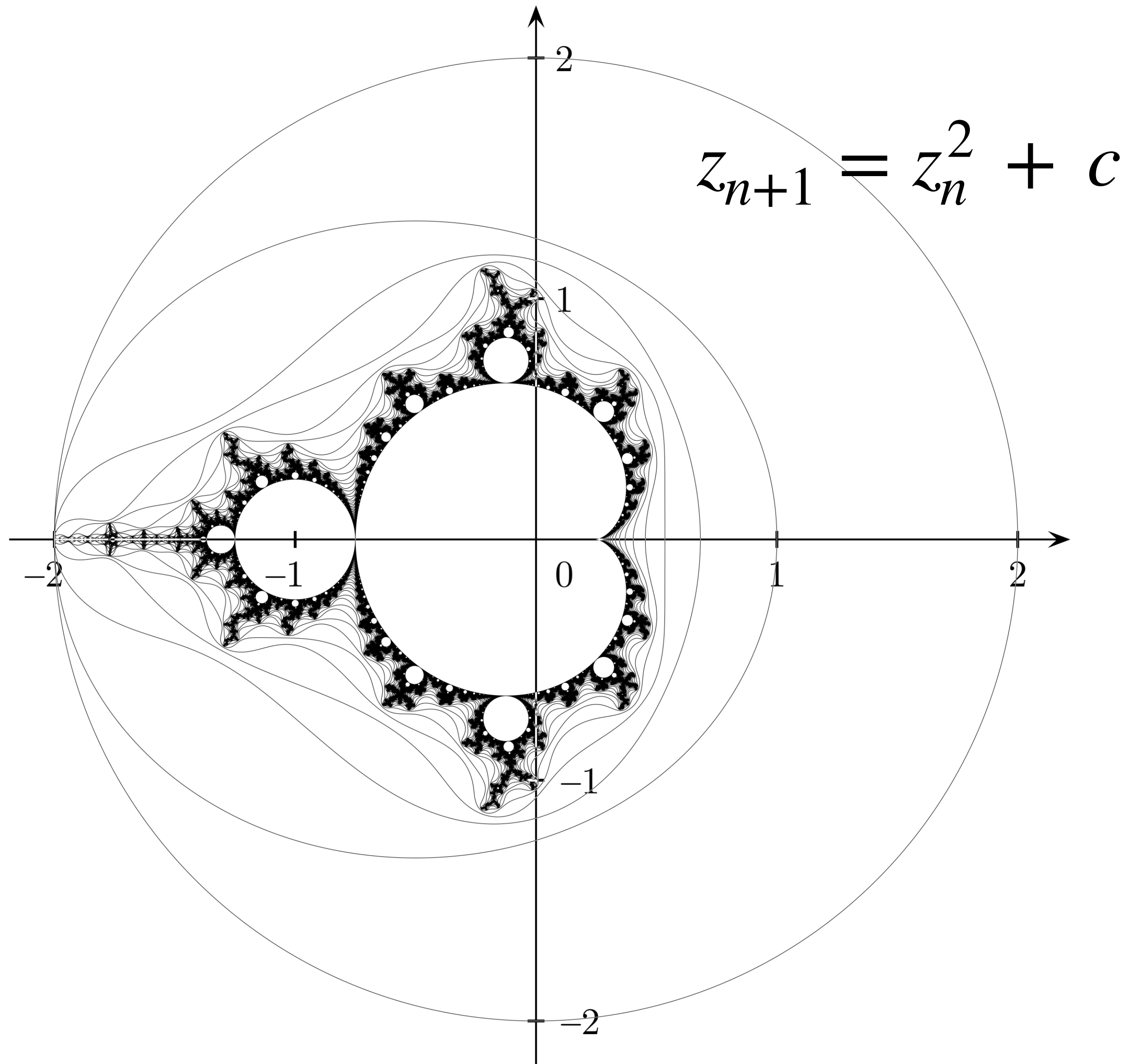


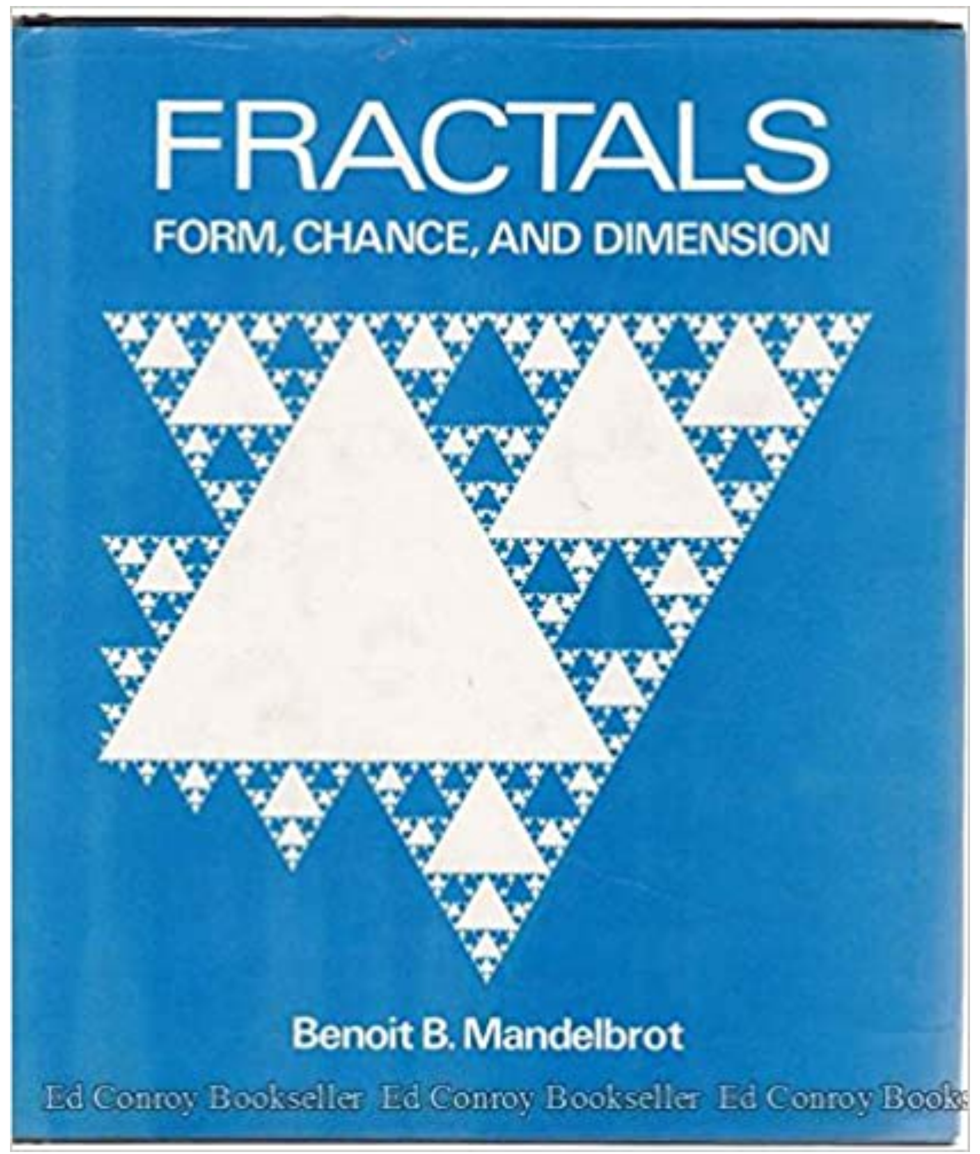
[Hausdorff (1918), Mandelbrot (1967)]

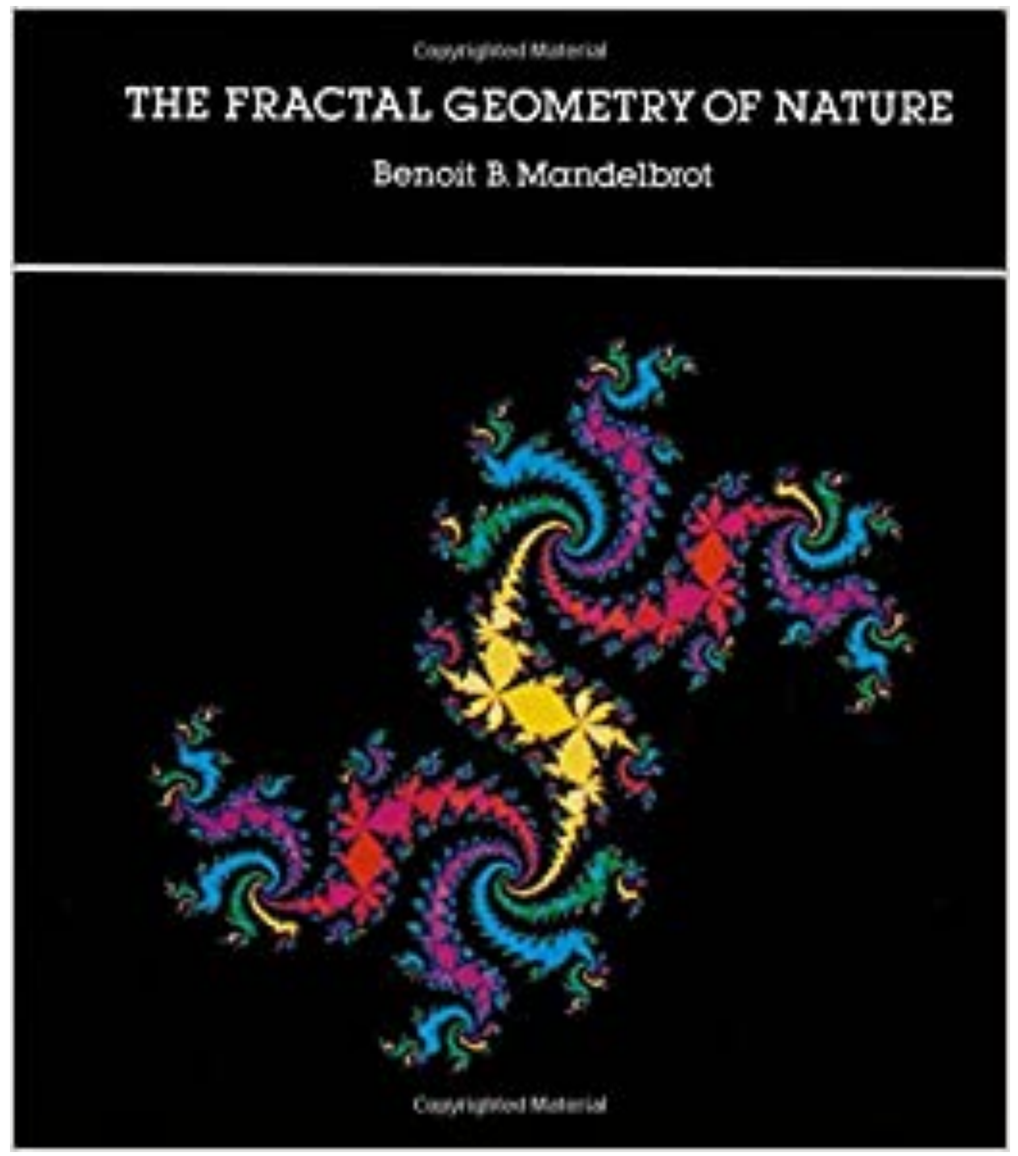
The father of fractals: Benoit Mandelbrot

20 November 1924 – 14 October 2010

* He was an expert of "**the art of roughness**" of physical phenomena
* He is recognized for his contribution to the field of fractal geometry
* He coined the word "**fractal**"
* He developed the theory of "**roughness and self-similarity**" in nature







...but the story starts before Mandelbrot

8 November 1868 - 26 January 1942

* one of the founders of modern topology

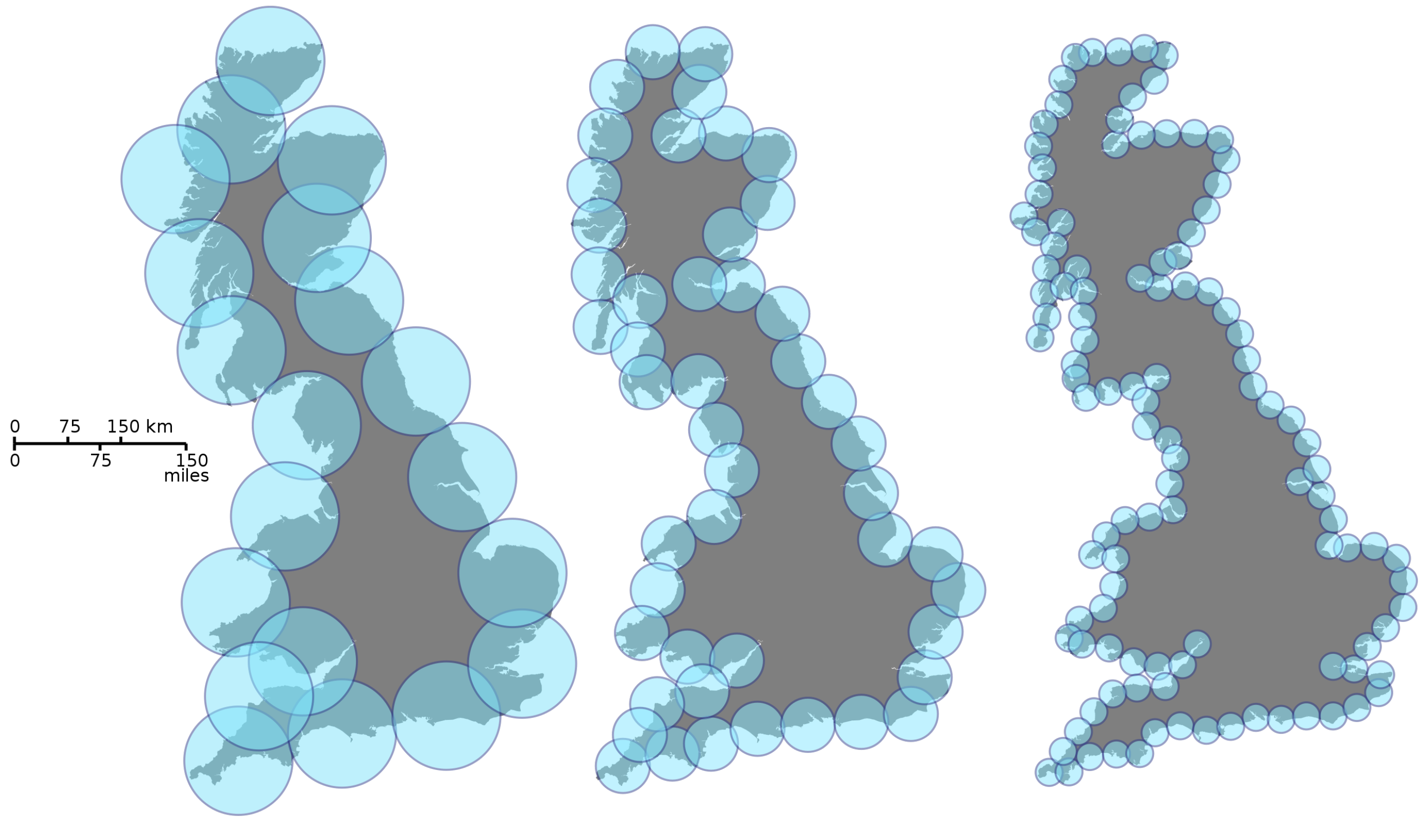
* contributed significantly to measure theory

* Hausdorff measure: covering the surface with "small-radius balls"

$$\mathcal{H}^d(S) := \liminf_{r\to 0}\left\{ \sum_i r_i^d : \text{ there is a cover of } S \text{ by balls with radii } 0 < r_i < r \right\}$$

* Hausdorff dimension: measuring "highly rugged quantities"

$$d_{\mathrm{H}}(S) := \inf\{d \geq 0 : \mathcal{H}^d(S) = 0\}$$



Published: March 1918

Dimension und äußeres Maß

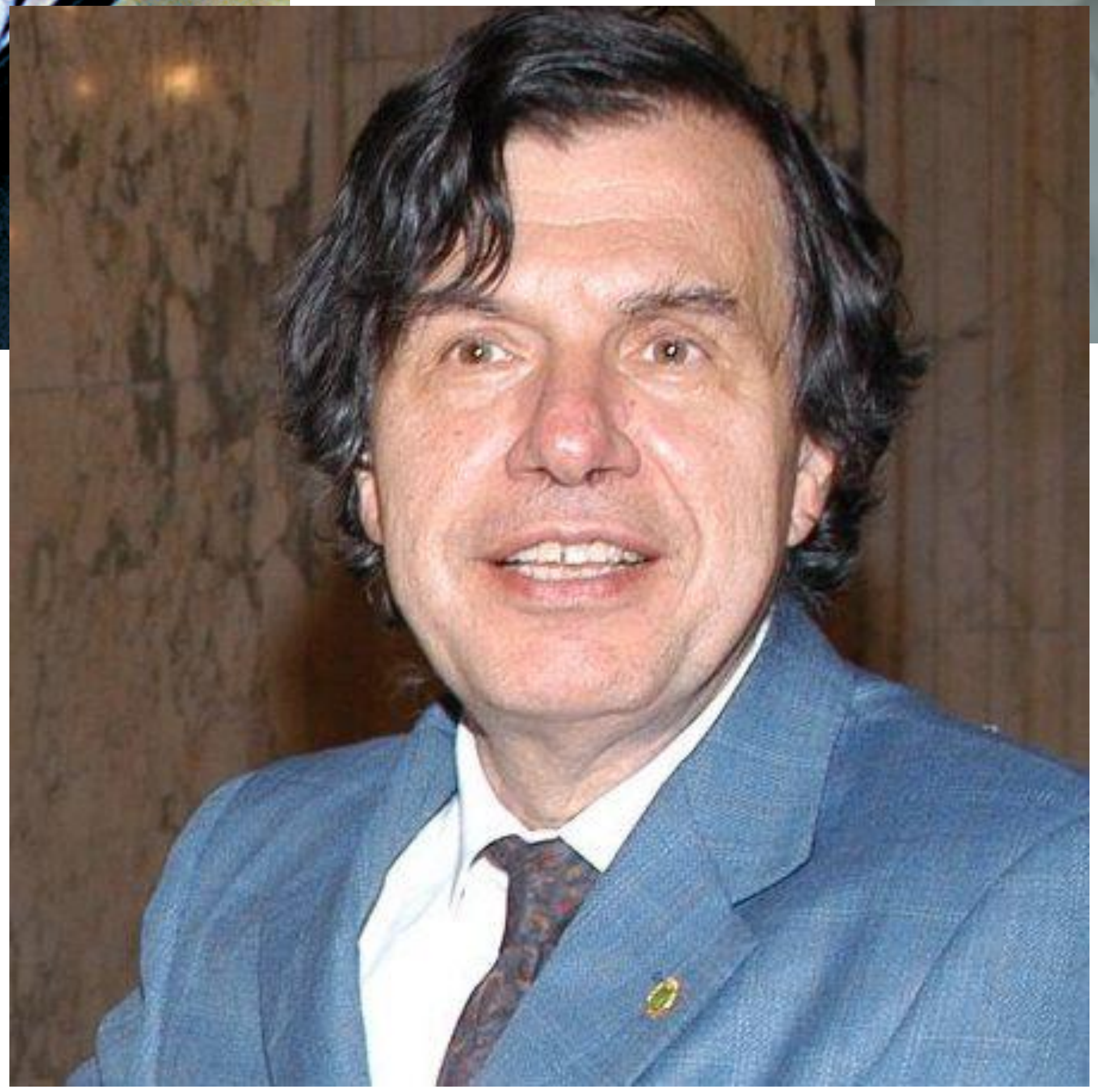
Felix Hausdorff

Mathematische Annalen **79**, 157–179(1918) | Cite this article

607 Accesses | **629** Citations | **3** Altmetric | Metrics

...and continued soon after Mandelbrot

Itamar Procaccia



Giorgio Parisi

Uriel Frisch

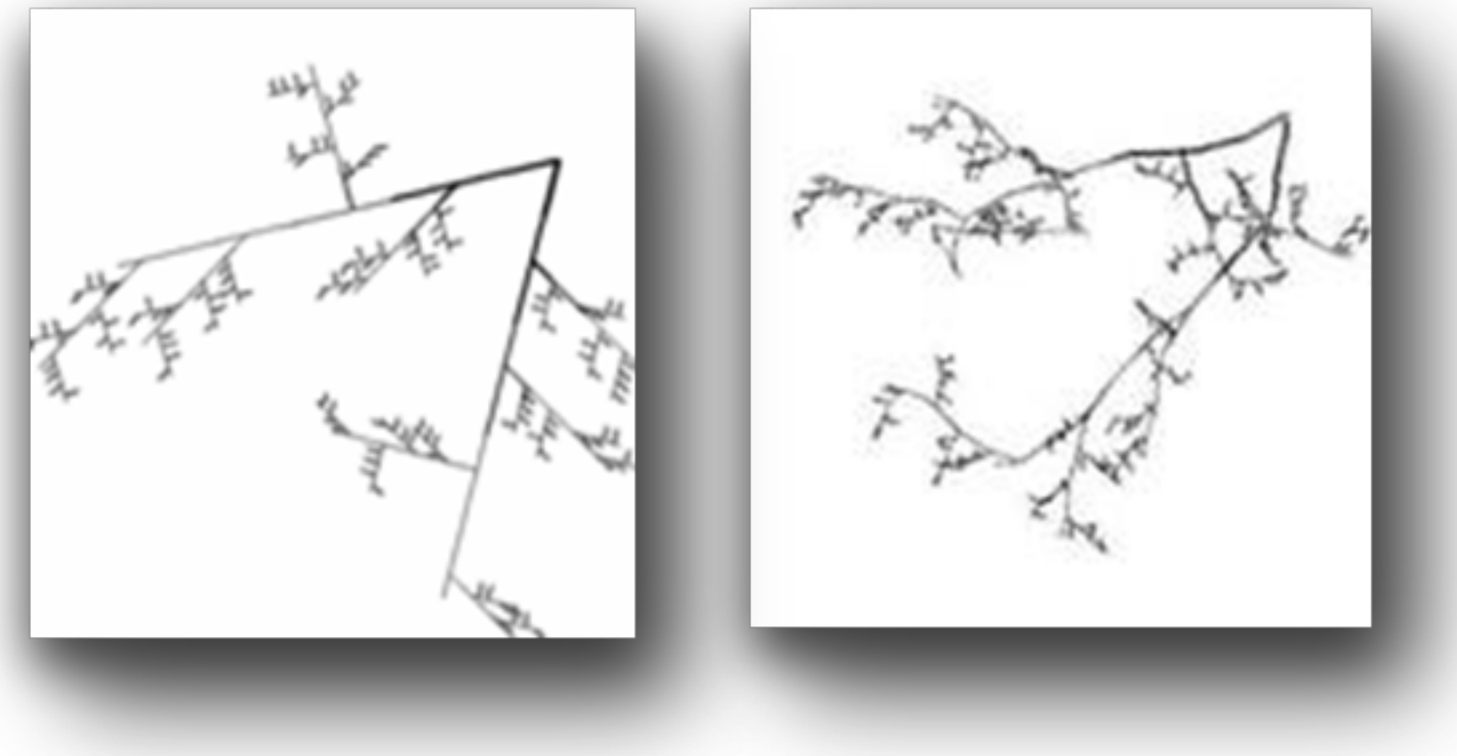
Peter Grassberger

Roberto Benzi

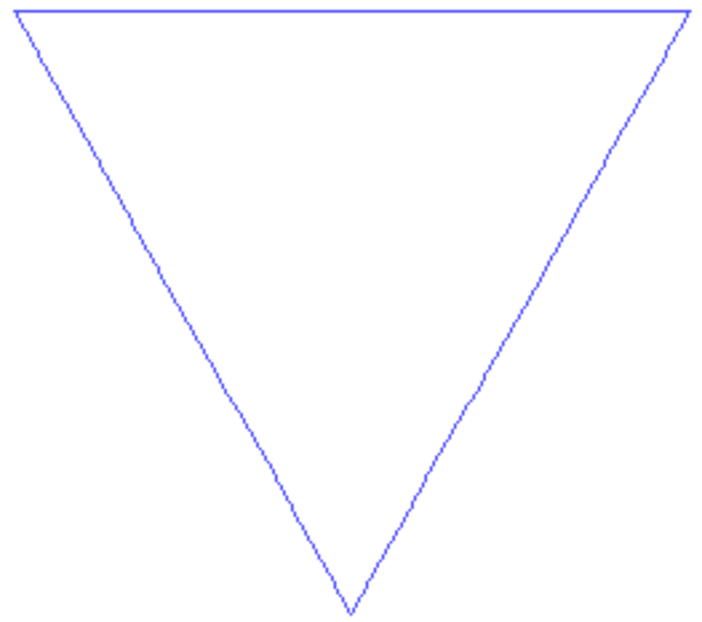
Is a single exponent enough to describe its dynamics?

d_H quantifies the **inherent scaling** but does not uniquely provide enough **information to reconstruct it**

Rule: produce 4 new parts for every 1/3 scaling $\longrightarrow$ $$d_H = \frac{\log N}{\log k} = \frac{\log 4}{\log 3} \approx 1.26\ldots$$



Fractal tree

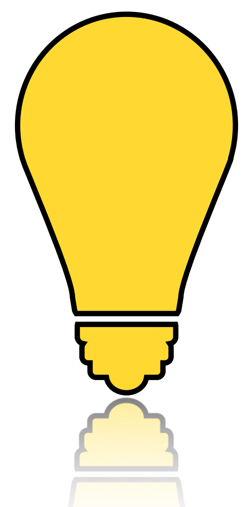


Koch snowflake

Same rule

Same scaling

Drastically different!



$d_H \rightarrow D_q$

The generalized fractal dimensions D_q

introduced in 1980s both in the framework of dynamical systems and fully developed turbulence

Dynamical systems

* let $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \{\beta\})$ be a continuous dynamical system

* let $\mathscr{M}$ the invariant set of its phase-space

* let $\mathscr{S}$ a partition of $\mathscr{M}$ into $\mathscr{B}_i$ boxes or hypercubes

* let $\mathscr{P}_i = \dfrac{\mu(\mathscr{B}_i)}{\mu(\mathscr{M})}$ the normalized measure of this box

$$D_q = \frac{1}{q-1} \lim_{r \to 0} \frac{\log \sum_i \mathscr{P}_i^q}{\log r}$$

[Hentschel and Procaccia (1983), Grassberger and Procaccia (1983a), Halsey+ (1986)]

Fully developed turbulence

* let $s(\vec{r})$ a physical system and let $\vec{a}$ a scale parameter

* let assume that $s(\vec{x} + \vec{a}) - s(\vec{x}) \sim a^{h(\vec{x})}$

* $h(\vec{x})$ quantifies local degree of singularity around $\vec{x}$

* let $\mathscr{I}$ the set of points $\vec{x}$ sharing the same $h(\vec{x})$

$$D(h) := \inf\{d \geq 0 : \mathscr{I}^d(\vec{x}) = 0\}$$

[Parisi and Frisch (1985)]

The generalized fractal dimensions D_q

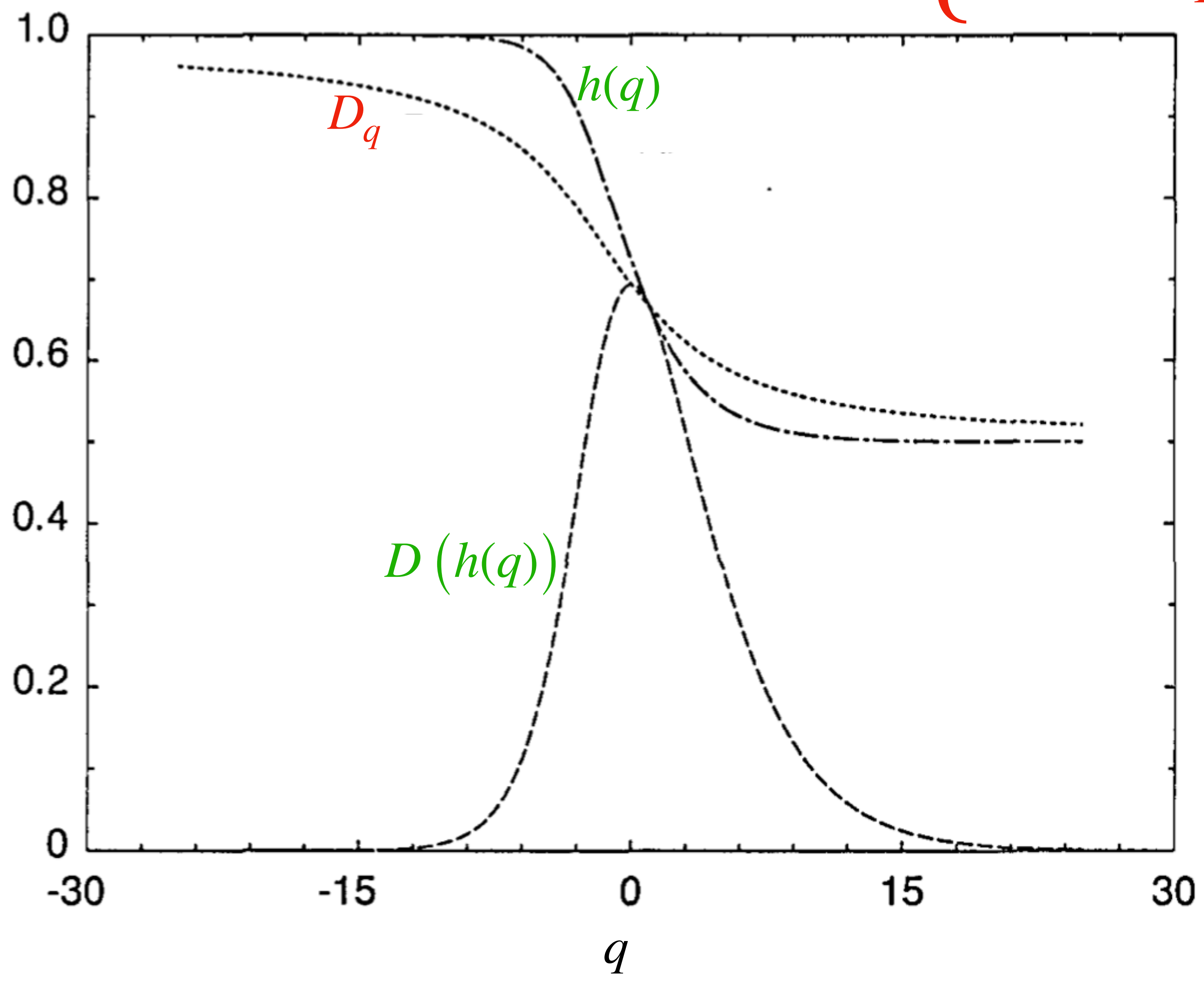
apparently the two framework are independent and not connected but...

[Hentschel and Procaccia (1983)]

Legendre Transform: $\mathscr{L}_n\{f(x)\} = \tilde{f}(n) = \int_{-1}^{1} P_n(x)\, f(x)\; dx$

$$\big(h, D(h)\big) \to (q, D_q)$$

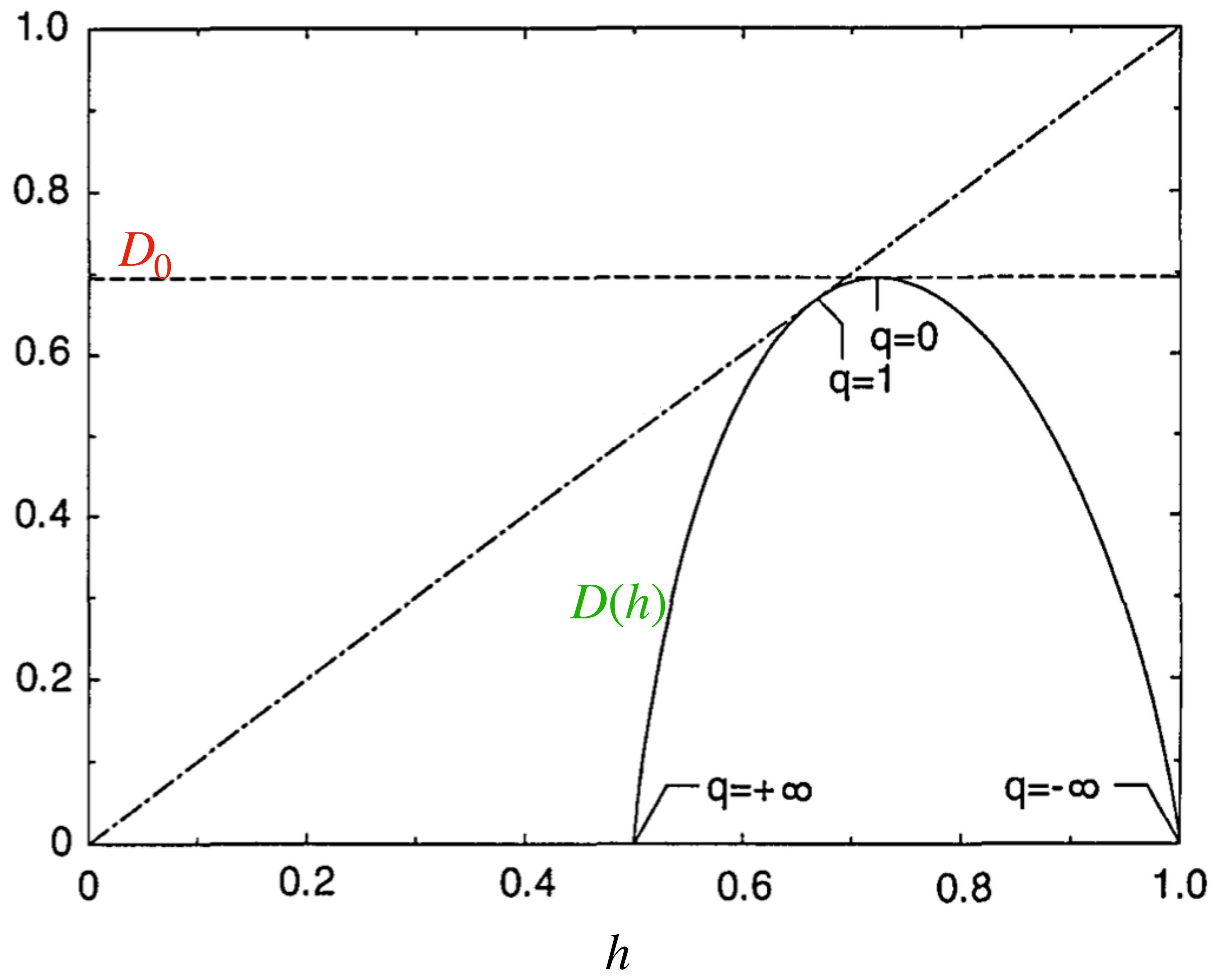
$$\begin{cases} q = \dfrac{dD(h)}{dh} \\ D_q = \dfrac{1}{q-1}\big(qh - D(h)\big) \end{cases} \qquad \begin{cases} h = (q-1)\dfrac{dD_q}{dq} \\ D(h) = hq - (q-1)D_q \end{cases}$$



$$D_q < D_{q'} \quad if \quad q > q'$$

$$D_0 = \max_h \{D(h)\}$$

$$D(h) = 0 \quad for \quad q = \pm\infty$$



D_0, D_1, and D_2

Box counting dimension

suppose that $N(\varepsilon)$ is the number of boxes of side length ε required to cover the set, then

$$D_0 = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \frac{1}{\varepsilon}}$$

[Hentschel and Procaccia (1983)]

Information dimension

how the average information needed to identify an occupied box scales with box size

$$D_1 = \lim_{\varepsilon \to 0} \frac{-\langle \log p_\varepsilon \rangle}{\log \frac{1}{\varepsilon}}$$

[Hentschel and Procaccia (1983)]

Correlation dimension

let M be the number of phase-space points and let g_ε be the number of pairs of points closer than ε to each other

$$D_2 = \lim_{M \to \infty} \lim_{\varepsilon \to 0} \frac{\log(g_\varepsilon / M^2)}{\log \varepsilon}$$

[Grassberger and Procaccia (1983a)]

The multiscale generalized fractal dimensions $D_{q,\tau}$

D_q only allows a "global" view of the system, how to consider the multiscale nature of complex systems?

* a complex system is formed by many interacting components at different scales

$$x(t) = \langle x(t) \rangle + \delta x_{\tau_1}(t) + \ldots + \delta x_{\tau_k}(t) = x_0 + \sum_{\tau} \delta x_{\tau}(t)$$

* let be $\mathscr{M}_\tau$ the phase-space representative of the system for all scales $\tau' < \tau$

* let $\mathscr{S}_\tau$ a partition of $\mathscr{M}_\tau$ into $\mathscr{B}_{i,\tau}$ boxes or hypercubes of size r

* let $\mathscr{P}_{i,\tau} = \dfrac{\mu(\mathscr{B}_{i,\tau})}{\mu(\mathscr{M}_\tau)}$ the normalized measure of this box

$$D_{q,\tau} = \frac{1}{q-1} \lim_{r \to 0} \frac{\log \sum_i \mathscr{P}_{i,\tau}{}^q}{\log r}$$

- scale-dependent complexity measures
- local properties of fluctuations
- convergence: $\tau' \to \tau$ then $D_{q,\tau} \to D_q$

[Alberti+ (2020)]

D. Chaotic measures: Lyapunov spectrum and predictability

What is chaos?

When the present determines the future, but the approximate present does not approximately determine the future

Chaotic systems: apparently random states governed by underlying patterns and deterministic laws that are highly sensitive to initial conditions

↓

an arbitrarily small change of the current trajectory may lead to significantly different future behavior

Properties of the chaotic systems

1. Topological mixing: the system evolves over time so that any given region or open set of its phase space eventually overlaps with any other given region
2. Dense of periodic orbit: every point in the space is approached arbitrarily closely by periodic orbits
3. Strange attractor: the subset of phase space at which a large set of initial conditions leads to orbits that converge to this chaotic region

   3.1 Poincaré-Bendixson theorem: for continuous dynamical systems a strange attractor can only arise in three or more dimensions

   3.2 Finite-dimensional linear systems are never chaotic; for a dynamical system to display chaotic behavior, it must be either nonlinear or infinite-dimensional

[Ott (2002), Strogatz (2014)]

The "control" of chaos...

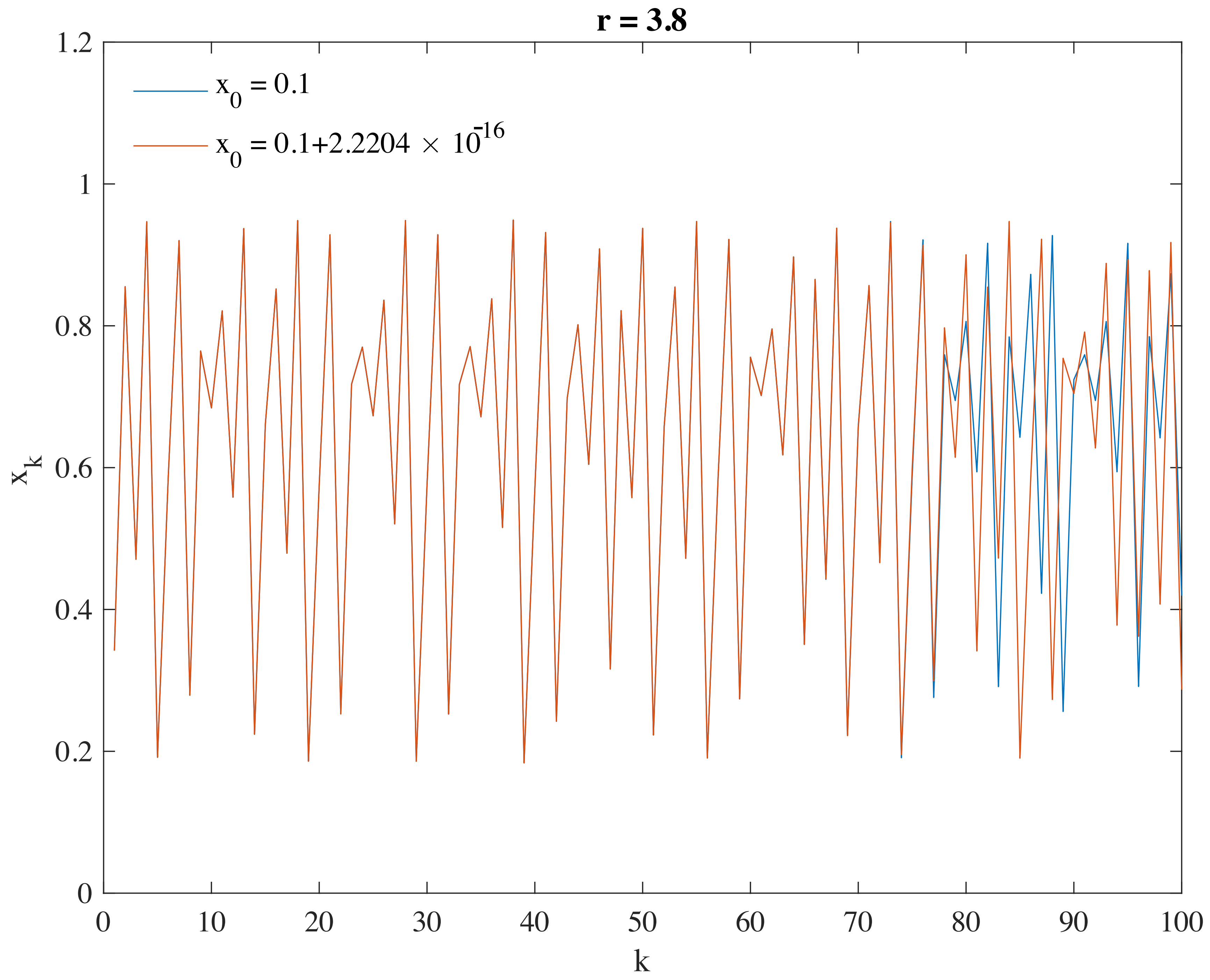
Usually the trajectory of chaotic systems is dependent on **control** or **bifurcation parameters**

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k, \beta_k) \quad \vee \quad \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \{\beta\})$$

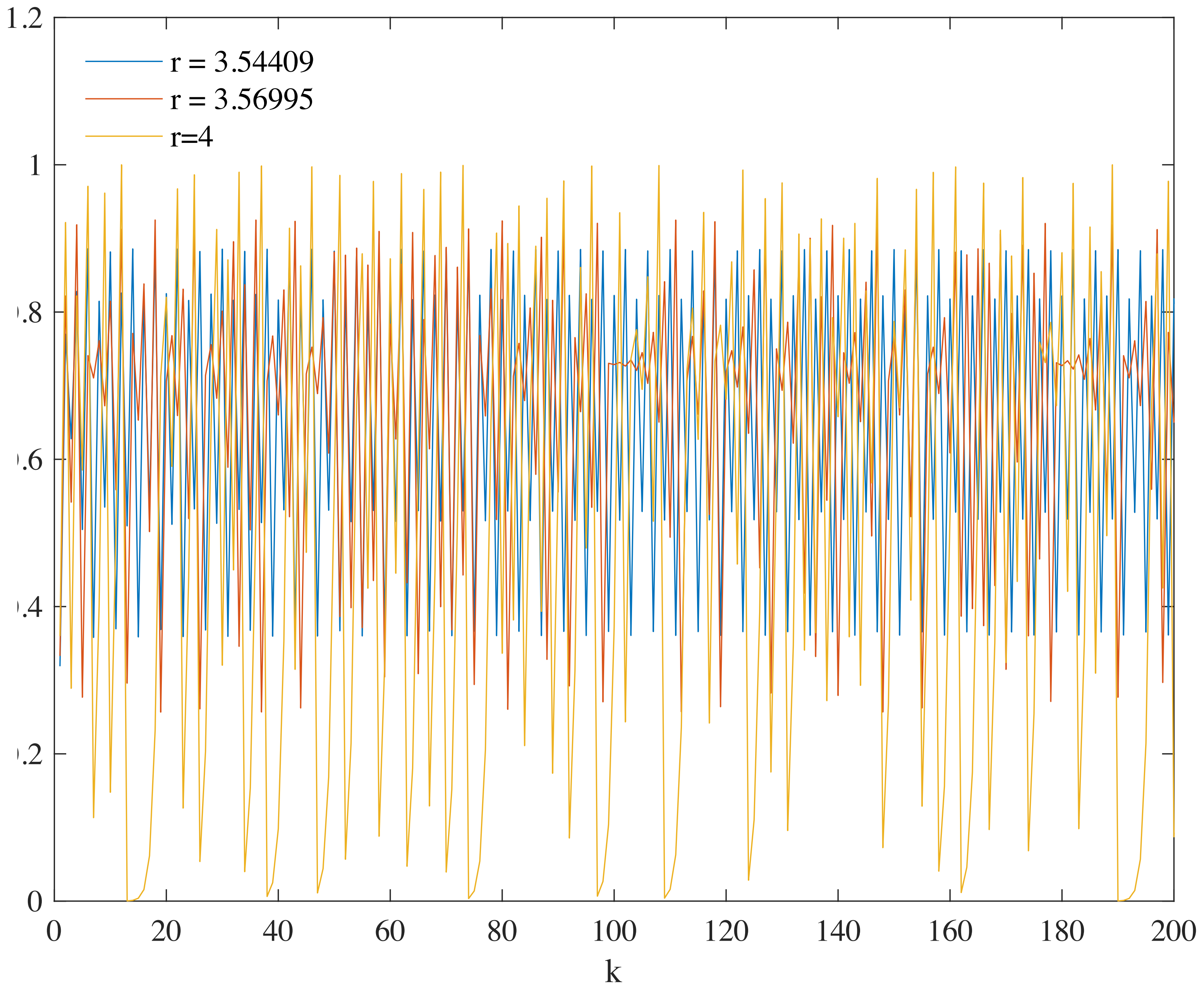
A famous simple example: the **logistic map**

$$x_{k+1} = r\, x_k\,(1 - x_k)$$

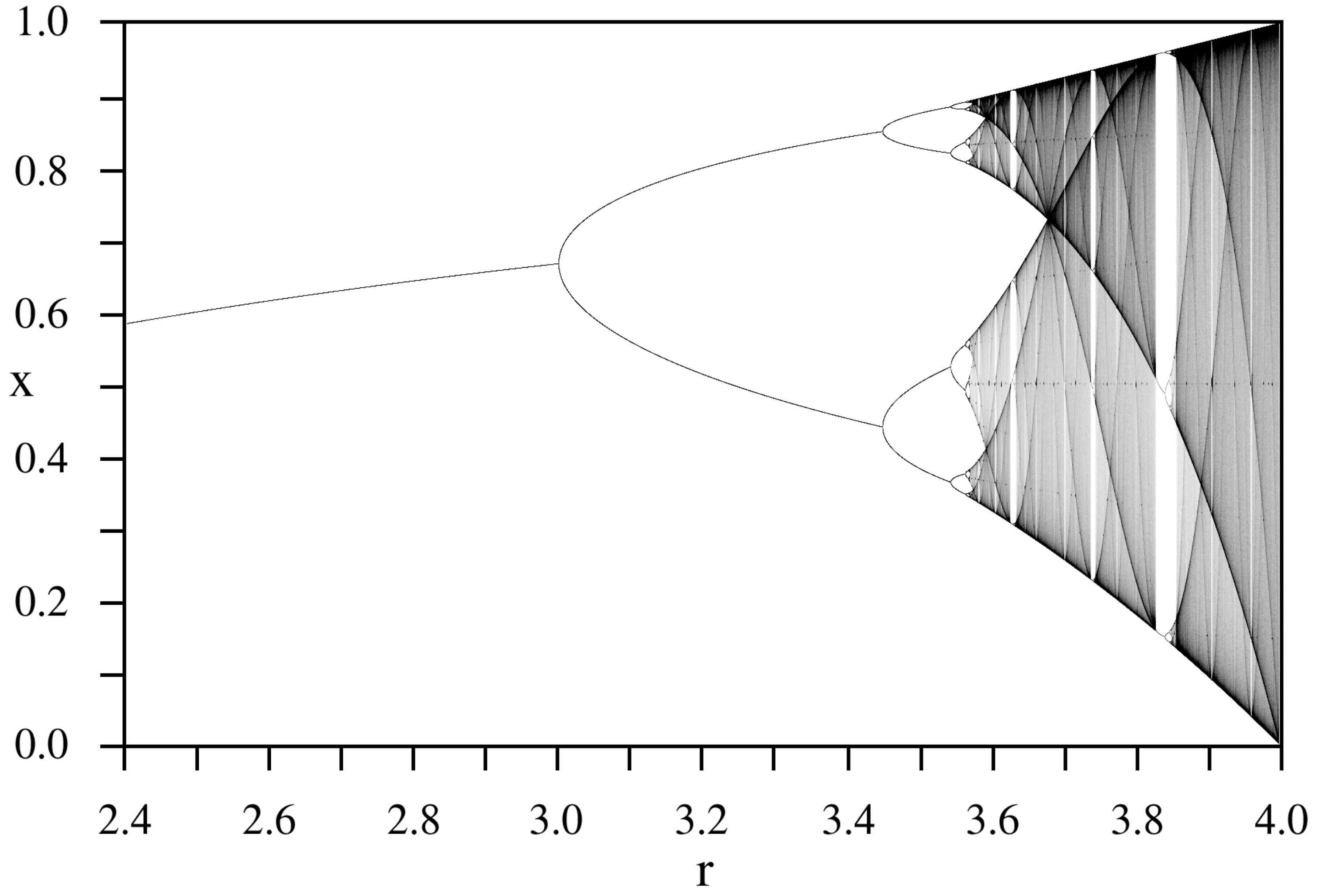
Dependence on initial conditions



The role of r



The bifurcation diagram



[Ott (2002), Strogatz (2014)]

Lyapunov exponents

a quantity that characterizes the rate of separation of infinitesimally close trajectories

* Two trajectories in the phase-space with initial separation vector $\delta \mathbf{Z}_0$ diverge at a rate given by

$$|\delta \mathbf{Z}(t)| \approx e^{\lambda t} |\delta \mathbf{Z}_0|$$

λ is the Lyapunov exponent

* There is a "spectrum" of Lyapunov exponents $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ as the dimension N of the system

* The Maximal Lyapunov exponent (MLE) determines a notion of predictability for a dynamical system

* A positive MLE is usually taken as an indication that the system is chaotic

* let k be the maximum integer such that the sum of the k largest exponents is still non-negative

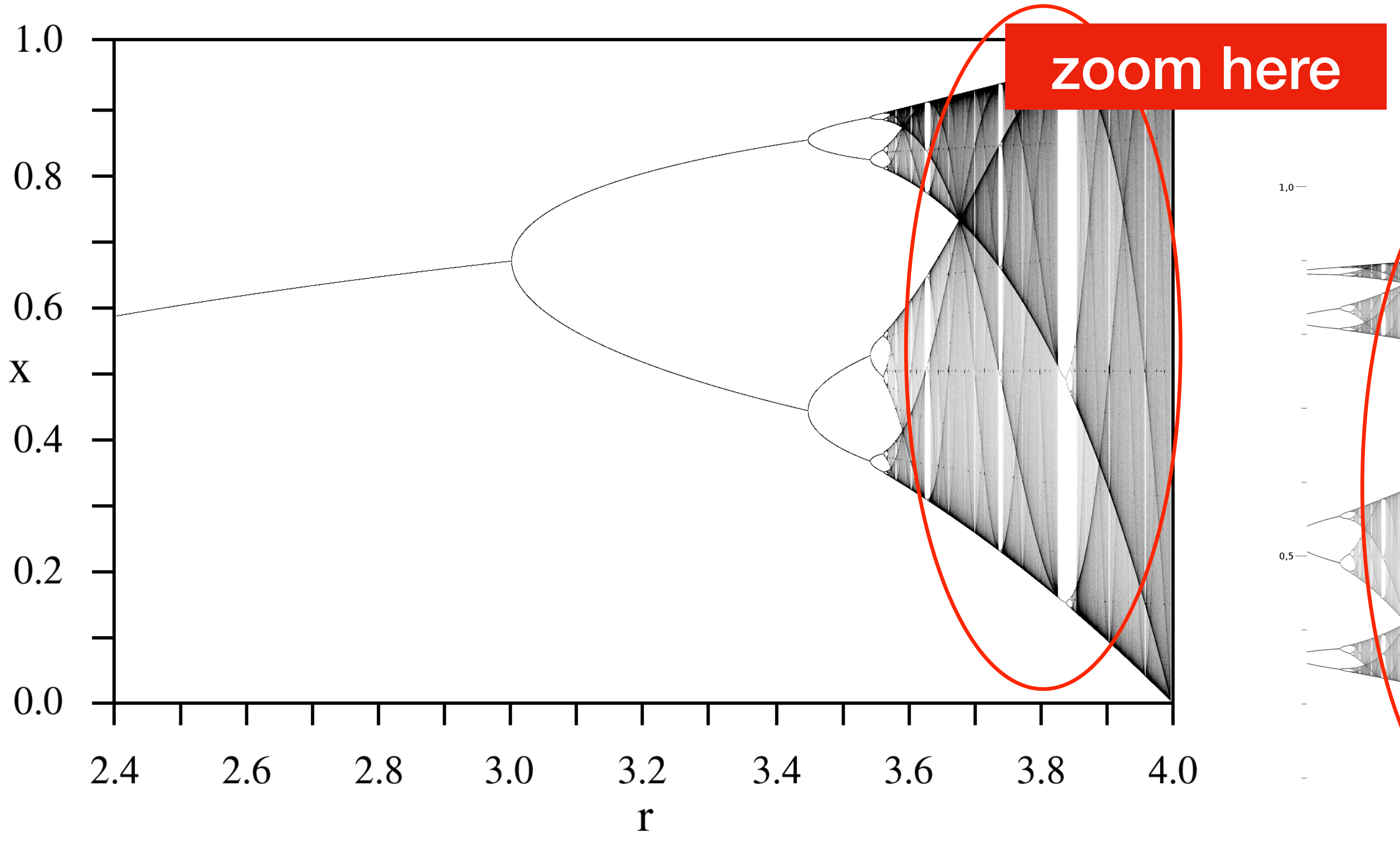
$$D_{KY} = k + \sum_{i=1}^{k} \frac{\lambda_i}{|\lambda_{k+1}|}$$

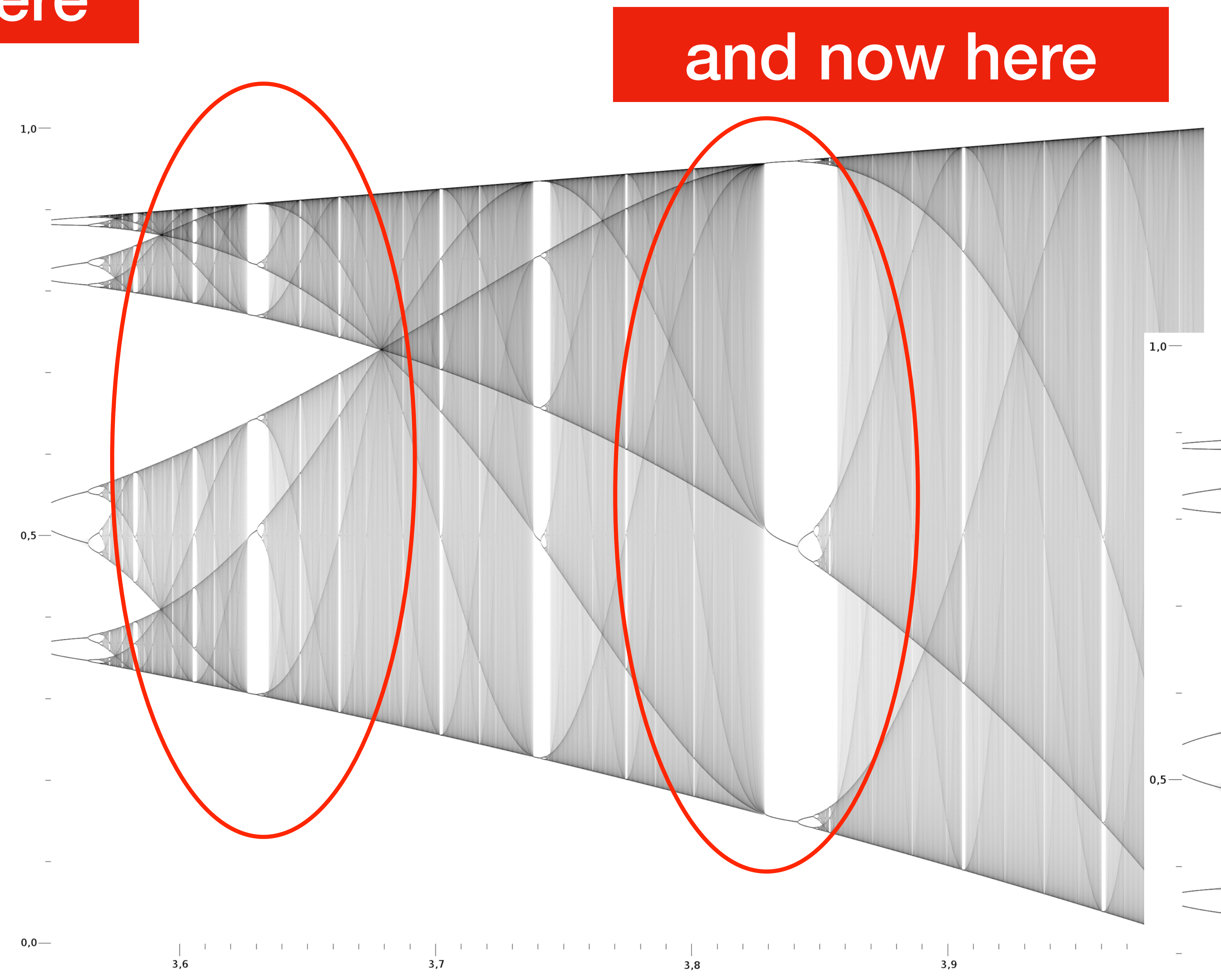
* D_{KY} is known as Kaplan-Yorke dimension and is an upper bound of D_1

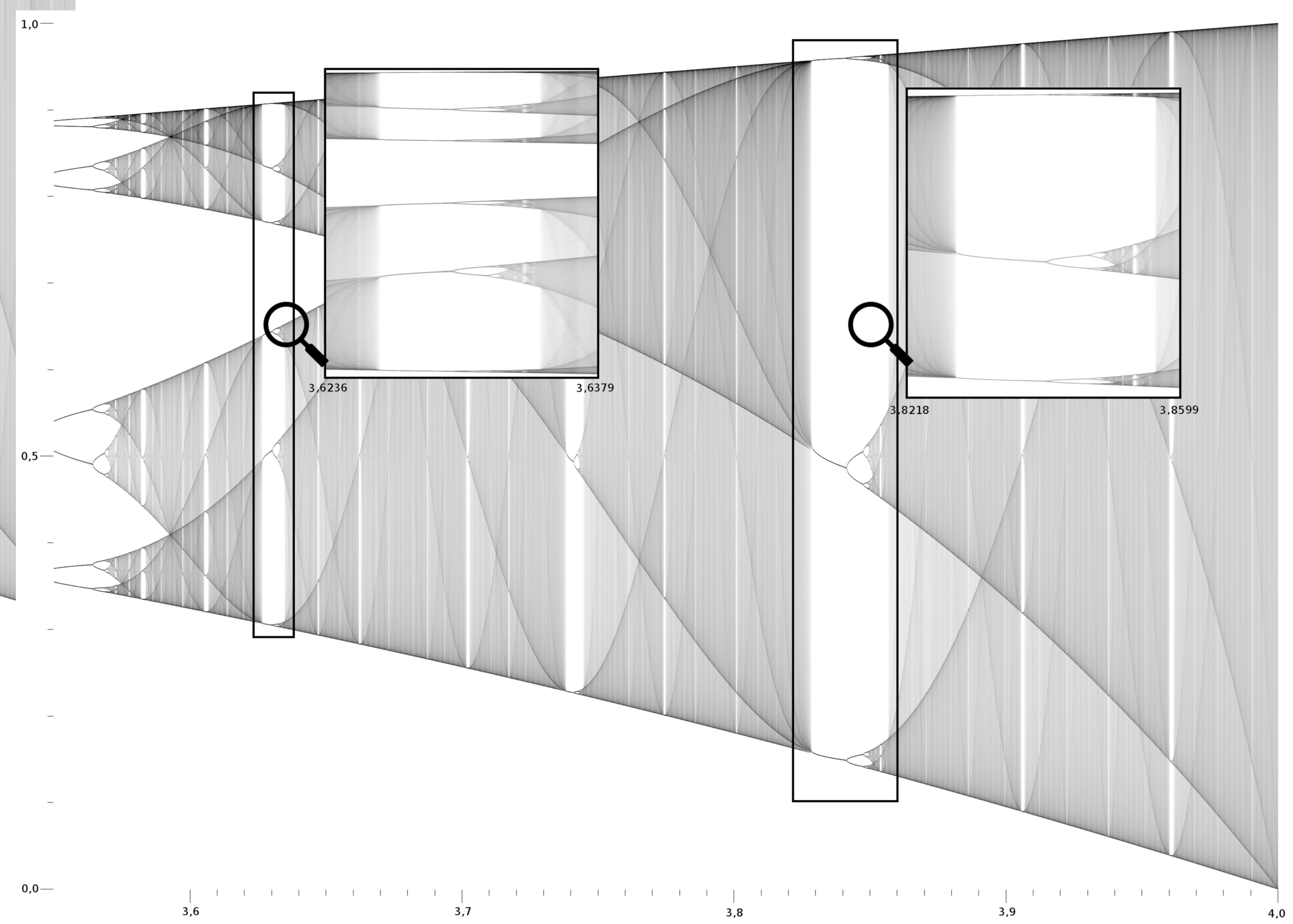
[Ott (2002), Strogatz (2014)]

Chaos meets fractals

Coming back again to the logistic map and its bifurcation diagram







It nearby looks like a **shrunk** and slightly **distorted** version of the whole diagram

[Ott (2002), Strogatz (2014)]

Kolmogorov entropy K2

a quantity that measures the predictability horizon of a given system

* Given a time series $f(t)$ we can reconstruct the embedded space via Takens theorem (1981)

$$\mathscr{M}[f(t)] = \mathbf{Y} = (f(t), f(t-\tau), \ldots, f(t-(m-1)\tau))^T$$

* We can define the correlation integral

$$C(r,m) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \Theta\left(r - \left|\mathbf{Y}_i - \mathbf{Y}_j\right|\right)$$

being N the number of phase-space states, Θ the Heaviside function, and r a threshold distance between two points in the phase-space

* In the limit of small r then $C(r,m)$ follow a power-law behavior: $C(r,m)\,|_{\lim r\to 0} \sim r^{D_2}$
* The Kolmogorov entropy is then defined as

[Grassberger and Procaccia (1983b)]

$$K_2 = \lim_{r\to 0} \frac{1}{\Delta t} \log \frac{C(r,m)}{C(r,m+1)}$$

* Forecast horizon: $\tau_2 = K_2^{-1}$
* Chaotic system: $K_2 < \infty$
* Non-deterministic system: $K_2 \to \infty$

E. Final remarks

Final remarks

1. No method is always and completely better/useful than another
2. Chaos is still everywhere, it is not needed to add other form of "disorder"
3. What you see "random" could not be always "noise"
4. Took from the past to make better the future...but always remember of the past!
5. Be curious for discovering and be sure for convincing

A friend said me: "If there is a process or a feature in Nature it shows without complicated and sometimes convoluted approaches"

Peter Ditlevsen

See you tomorrow for part II.

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