Fully developed turbulence: state of the art and new questions

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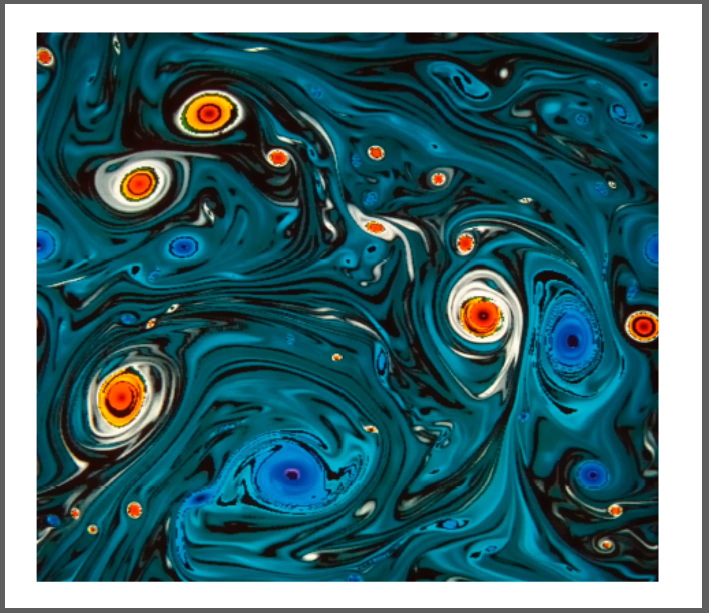


The talk is divided in two parts

In the first part, we consider a system (turbulence in three dimension). I will argue that, over the last 30 years, we have made enormous progress on many “open” questions. In particular we have developed a very powerful framework which unifies many different statistical properties observed in turbulent flows.

In the second part we look at turbulence from different points of view focusing on other general questions (rare or extreme events) and discussion how to use emergent methodology (machine learning) to look at the available informations in a different way.

FOCUS OF THE FIRST TALK:

Homogenous and Isotropic fully 3d fully developed turbulence (HIT)

This is a "narrow" view with respect to the large numbers of problems on turbulence.

(there is not "the problem" of turbulence)

Assumption: we consider a "simple" Newtonian incompressible fluid described by the Navier-Stokes equation

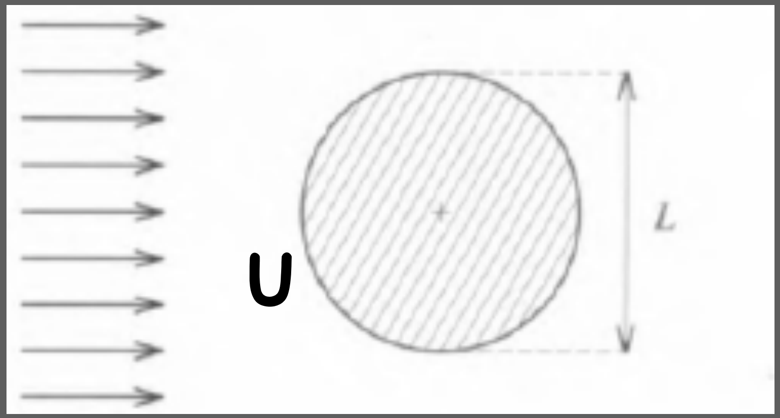
$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \boxed{\nu \Delta \mathbf{v}}$$

$$div \mathbf{v} = 0$$

Viscous effect (phenomenology)

$$\rho = const$$

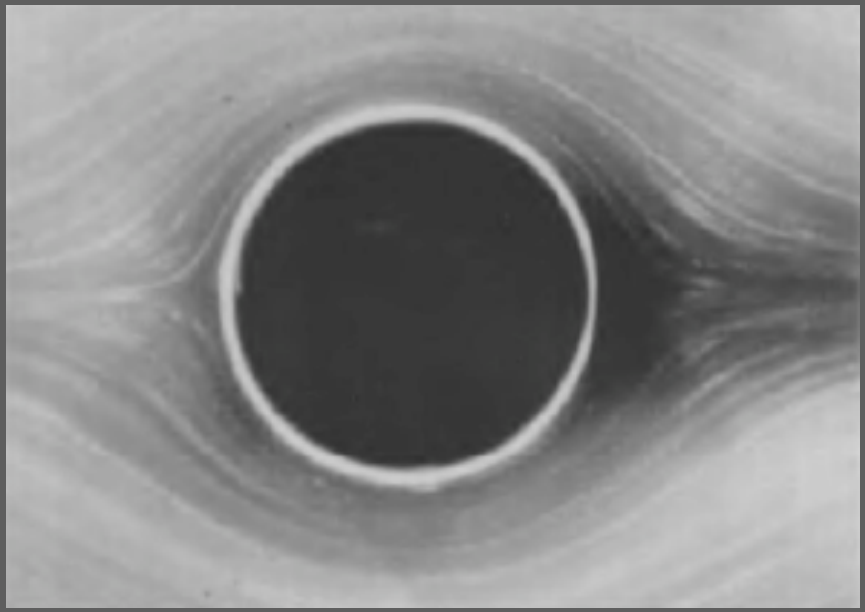
An important dimensional number: the Reynolds number



$$Re = \frac{UL}{\nu}$$

Re is the ratio of the characteristic time of viscous effect (L^2/ν) with respect to the characteristic time due to velocity advection (L/U)

Re = 0.16 Laminar Flow



Re = 1800 Turbulent flow

$$Re \to \infty \ corresponds\ to\ \nu \to 0$$

For a car moving at 100 km/h, Re is order 10^7

The general question we want to address concerns the statistical properties of velocity fluctuations in the limit $Re \to \infty$ (fully developed turbulence)

One question (among others): do the statistical properties of velocity fluctuations depend on the forcing mechanism and the viscous effects?

Universality with respect to:

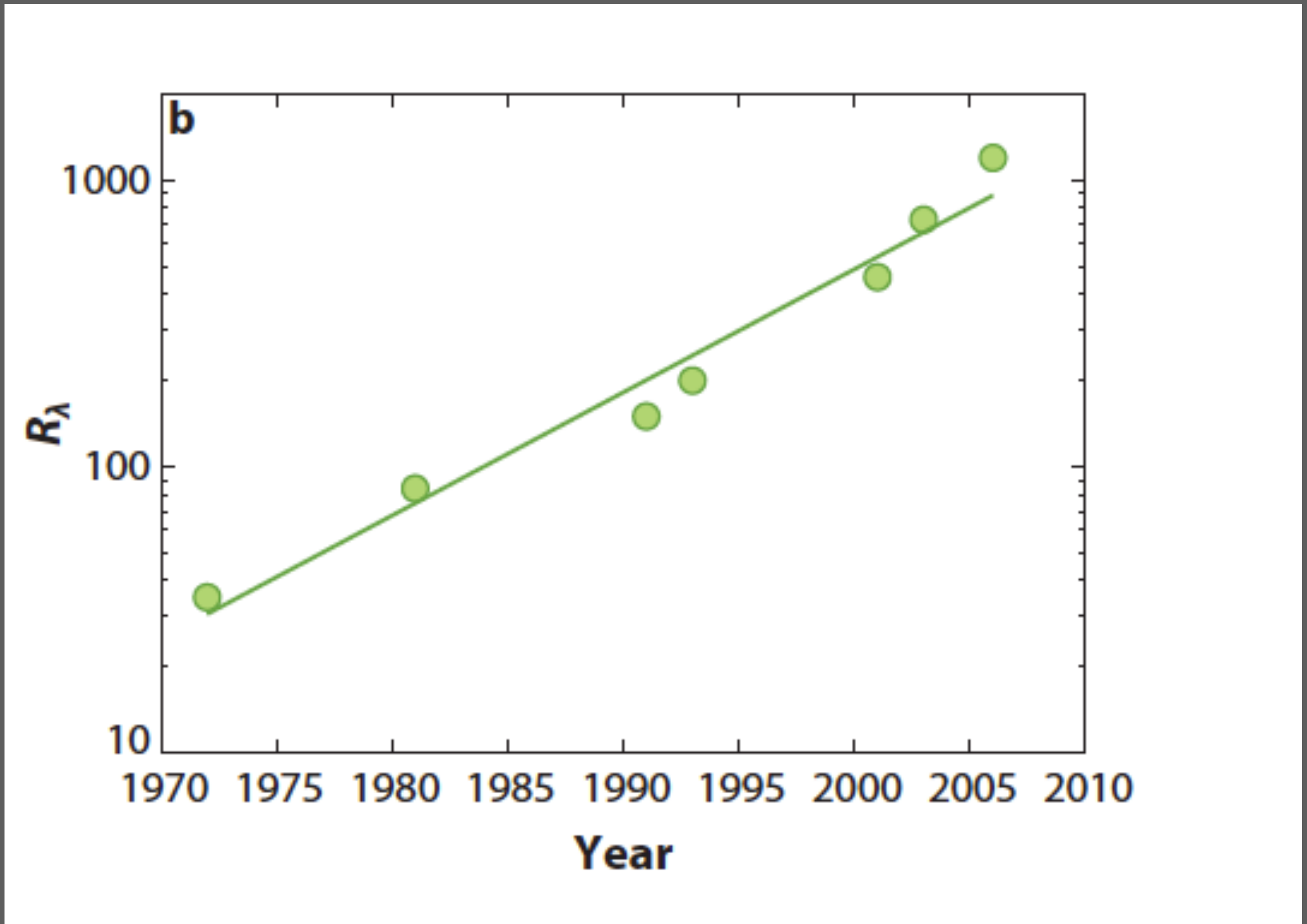
- forcing mechanisms
- dissipation mechanism

REMARK

We assume to know the equation of motion describing turbulence (Navier-Stokes equation) although the dissipation mechanism ($\nu \Delta \mathbf{v}$) is a "phenomenological" approximation.

Numerical simulations play an important role

$$Re = (Re_\lambda)^2$$



F Toschi 2009

We are now able to simulate turbulence flow with Re $\approx 10^7$
(close to the one achievable in laboratory experiments)

The basic property of fully developed turbulence

Energy dissipation in a turbulent flow is given by:

$$\epsilon = \nu\langle(\nabla \mathbf{v})^2\rangle$$

<...> = space/time average

The "zeroth law" of turbulence (viscous anomaly) states that:

ε is independent of Re for Re $\rightarrow\infty$

This is not trivial (not true for 2d turbulence) and it implies

$$\boxed{\nabla v \rightarrow Re^{1/2} \ for \ Re \rightarrow \infty}$$

Turbulence generates its own "ultraviolet" divergences!

First (historical) evidence of viscous anomaly: the Richardson diffusion

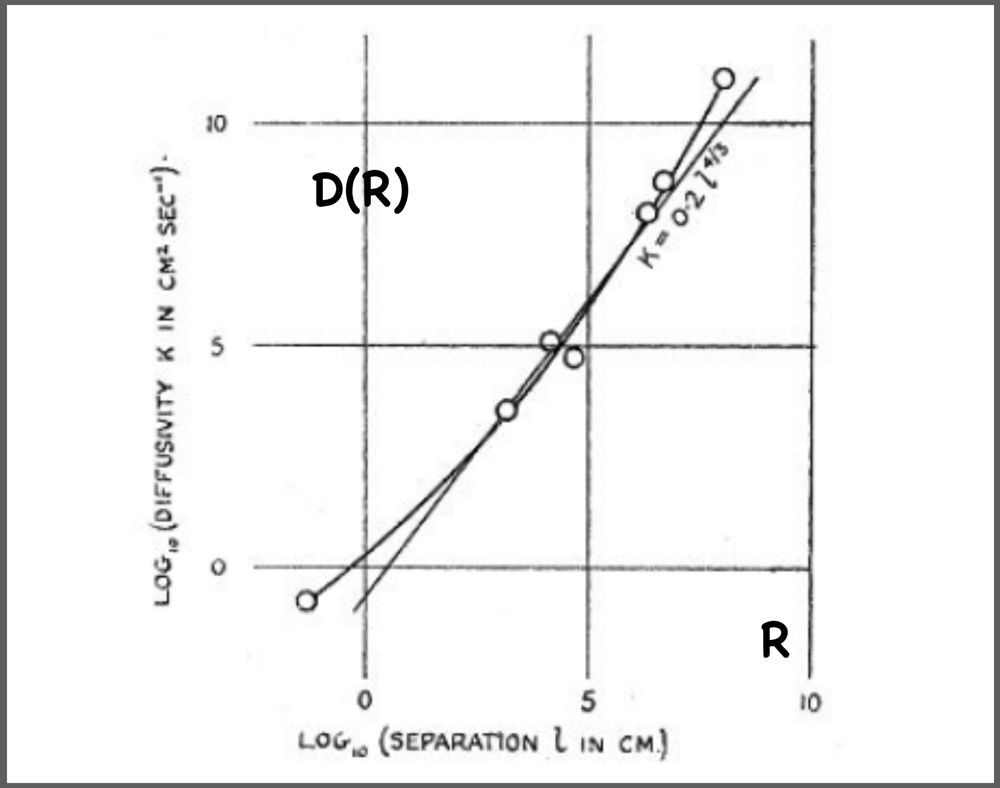
Idea: define P(R) as the probability distribution for 2 particles to be at distance R

$$\partial_t P = div[D(R)\nabla P]$$

$$D(R) = AR^{4/3}$$

Definition of turbulent diffusion

$$\frac{dR^2}{dt} = AR^{4/3}$$



From Richardson paper 1926

Consequences of Richardson Diffusion

Since $\frac{dR}{dt} \sim \delta U(R)$ (velocity difference) then $\frac{dR^2}{dt} \sim \delta U(R)\,R \equiv D(R) \sim R^{4/3}$

Thus Richardson diffusion $D \approx R^{4/3}$ implies that in the limit $Re \to \infty$ the velocity field is "rough" (Holder continuos):

$$\delta U(R) \sim R^{1/3}$$

The key observation (Gawedzki) is that Richardson diffusion leads to the breaking of Lagrangian trajectories in the limit of $Re \to \infty$.

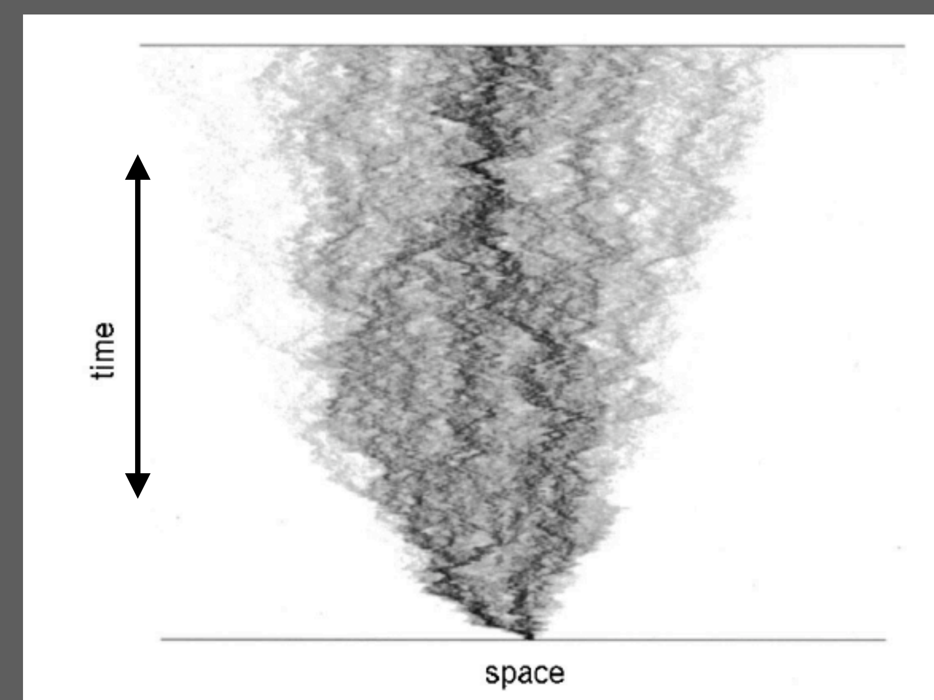
Breakdown of lagrangian trajectories

$$\frac{dR^2}{dt} = AR^{4/3} \rightarrow R^{2/3}(t) = R^{2/3}(0) + Bt$$

- Even for 2 particles starting with the same initial position, their distance growth in time
- 2 particles starting at distance R can collapse to the same position.

Historical remark: in the introduction to his paper Richardson wrote a paragraph

§ 1.2. *Does the Wind possess a Velocity?*



The Kolmogoroff 1941 theory for homogeneous and isotropic turbulence

$\delta v(r) \equiv (\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})) \bullet \frac{\mathbf{r}}{r}$ only relevant quantity for hom. iso. turbulence

$$\boxed{\langle (\delta v(r))^3 \rangle = -\frac{4}{5}\epsilon r + 6\nu \frac{d}{dr}\langle (\delta v(r))^2 \rangle}$$

Exact equation

ε is assumed to be independent on Re. Then for Re $\rightarrow \infty$ $\langle (\delta v(r))^3 \rangle = -\frac{4}{5}\epsilon r$

< $(\delta v(r))^3$> negative implies non linear energy transfer from large to small scales

Kolmogorov conjecture is that the probability distribution P[δv(r)] depends only on ε and r. Therefore

$\delta v(r) \sim \epsilon^{1/3} r^{1/3}$ agreement with Richardson findings

The Kolmogoroff 1941 theory for homogeneous and isotropic turbulence

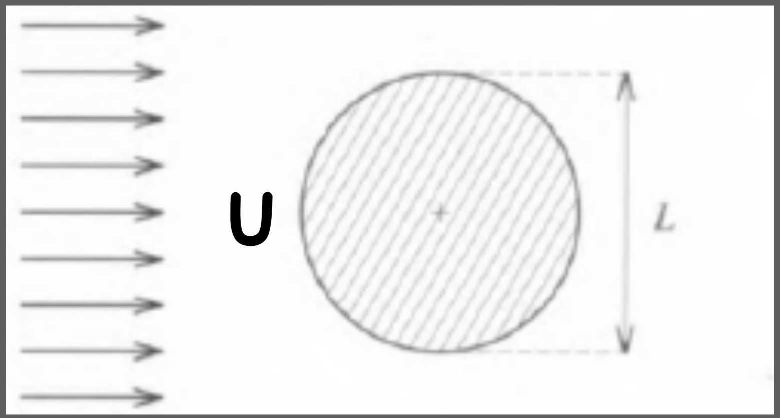
Viscous effect are relevant at scale η where

$$\epsilon\eta \sim 6\nu \frac{d}{dr}\langle(\delta v(r))^2\rangle|_{r=\eta} \to \eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim Re^{-3/4}$$

$$\boxed{\frac{\delta v(\eta)\eta}{\nu} \sim 1}$$

$$\tau_\eta = \frac{\eta}{\delta v(\eta)} \sim \left(\frac{\nu}{\epsilon}\right)^{1/2}$$

There exists a range of scales (**inertial range**) for $\eta \ll r \ll L$ (large scale of the system) where we should observe $\delta v(r) \approx r^{1/3}$. (energy spectrum $E(k) \approx k^{-5/3}$)

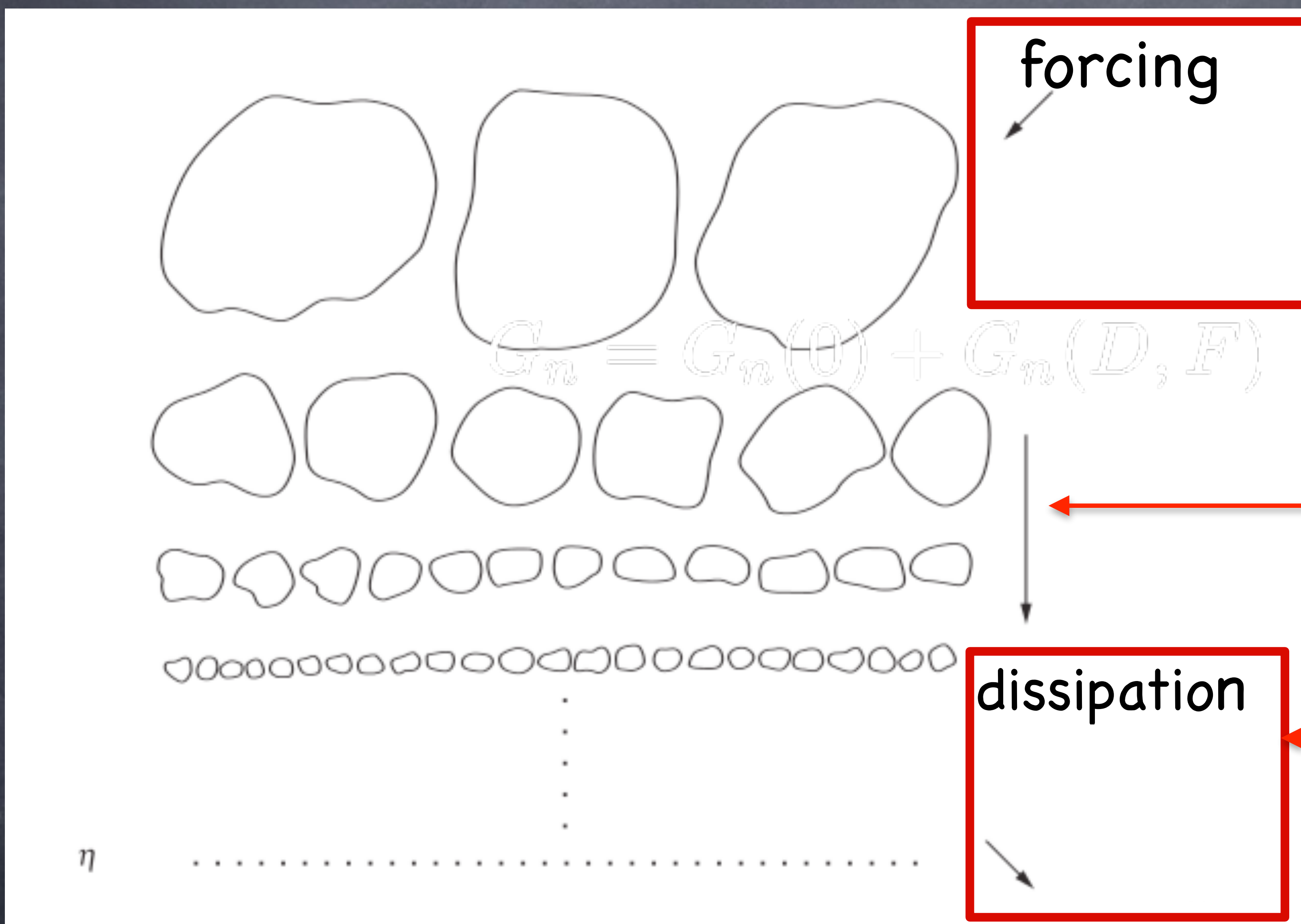


Example: for U=1m/sec L = 1m and $\nu=10^{-5}$ (air) we have

$\eta \approx 1$ mm $\quad \tau_\eta \approx 1$ mm sec

The Kolmogoroff 1941 equation for homogeneous and isotropic turbulence

$$\boxed{\langle(\delta v(r))^3\rangle = -\frac{4}{5}\epsilon r + 6\nu\frac{d}{dr}\langle(\delta v(r))^2\rangle}$$



cartoon

inertial range

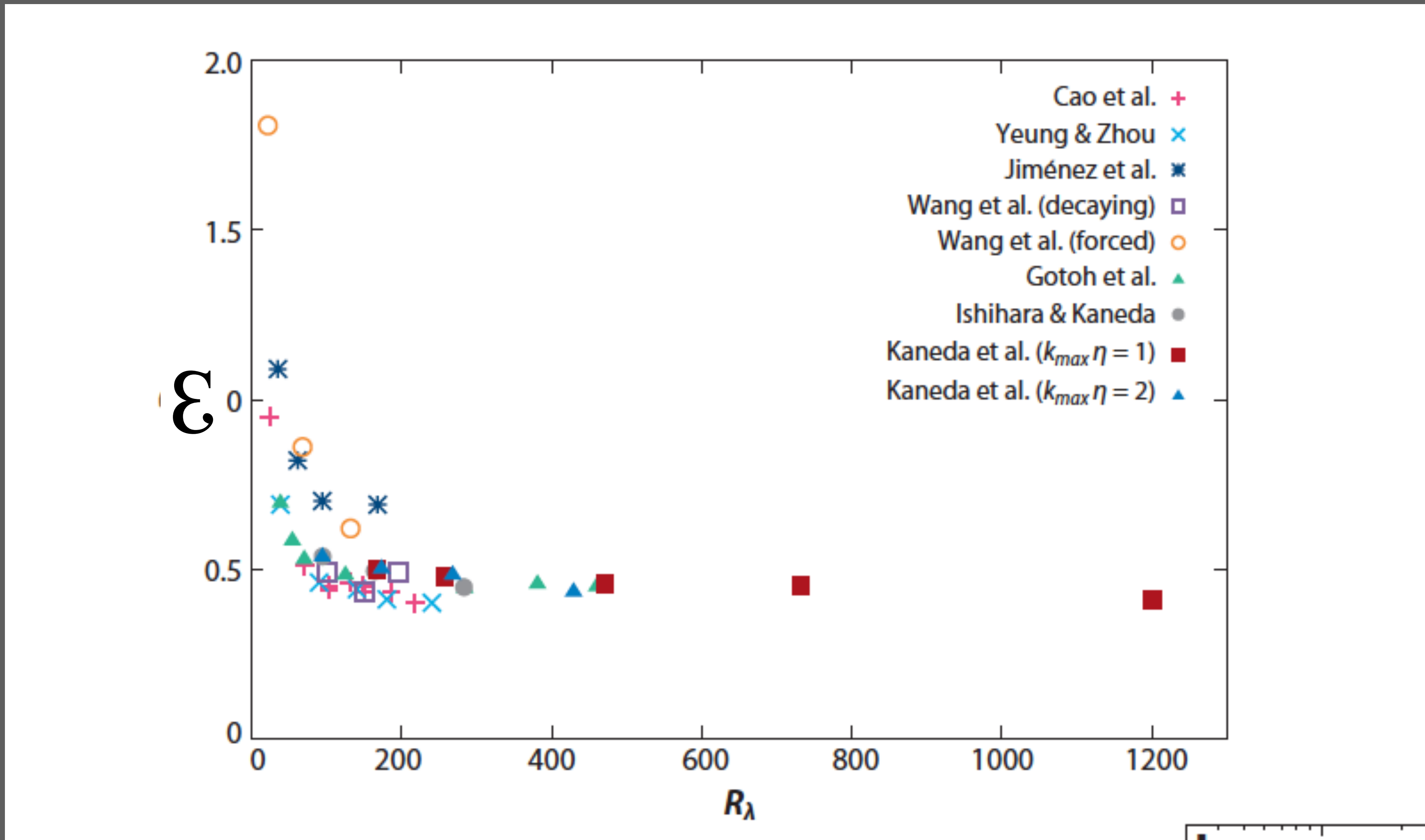
$\delta v(r) \approx \varepsilon^{1/3} r^{1/3}$

Dissipation scale η

$\delta v(r) \approx \eta \qquad \eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$

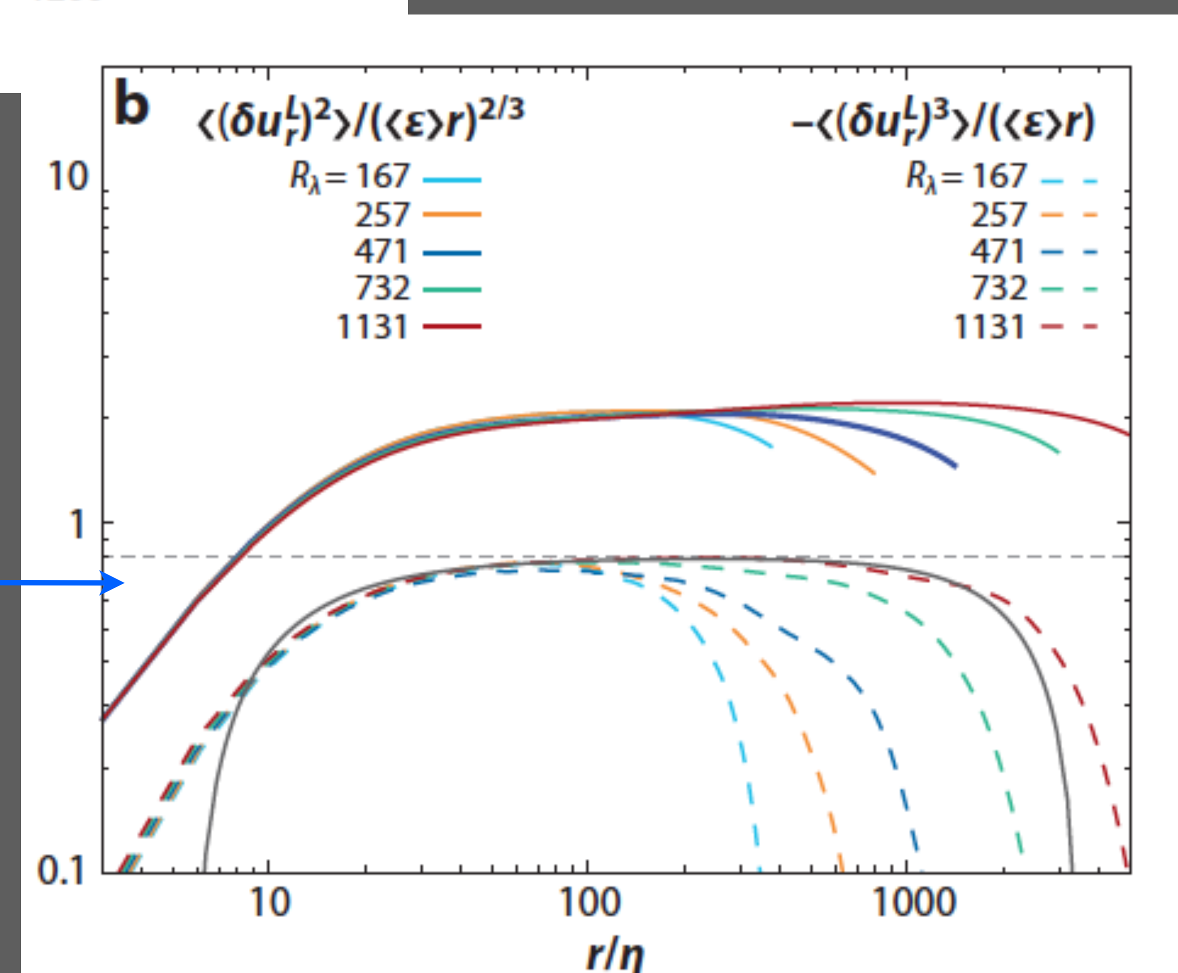
Kolmogorov 41 conjecture $P[\delta v(r)] = P[\varepsilon, r] \longrightarrow \delta v(r) \approx \varepsilon^{1/3}\ r^{1/3}$.

DNS results



$$\langle (\delta v(r))^3 \rangle = -\frac{4}{5}\epsilon r$$ is observed to be true

$\langle \delta v(r)^3 \rangle / r$



Ishihara,Gotoh,Kaneda 2009

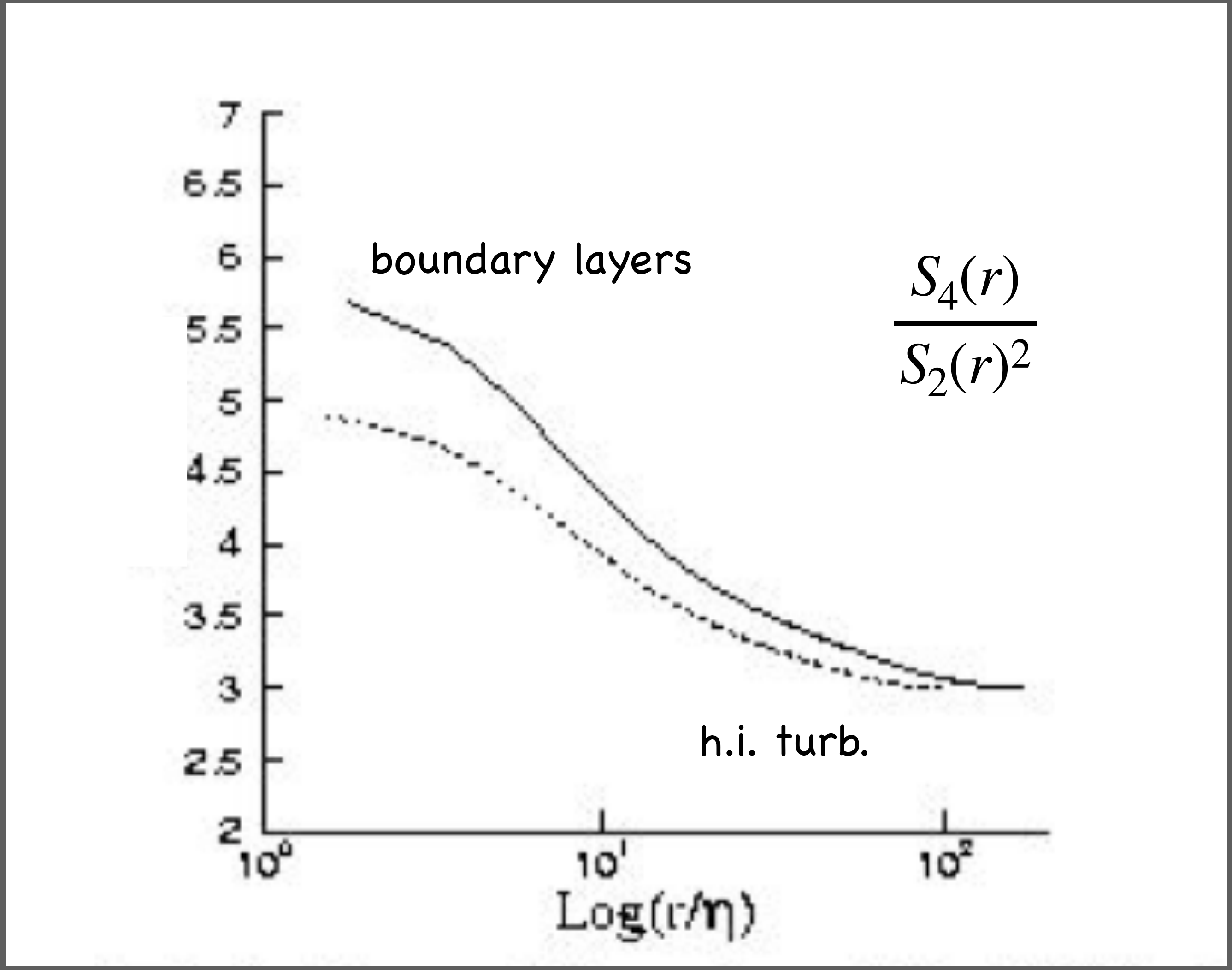
The Kolmogorov theory implies that the statistical properties of turbulence fluctuations are "universal" (provided r<<L for isotropy to hold) and are independent of the dissipation mechanism. Is that true ??

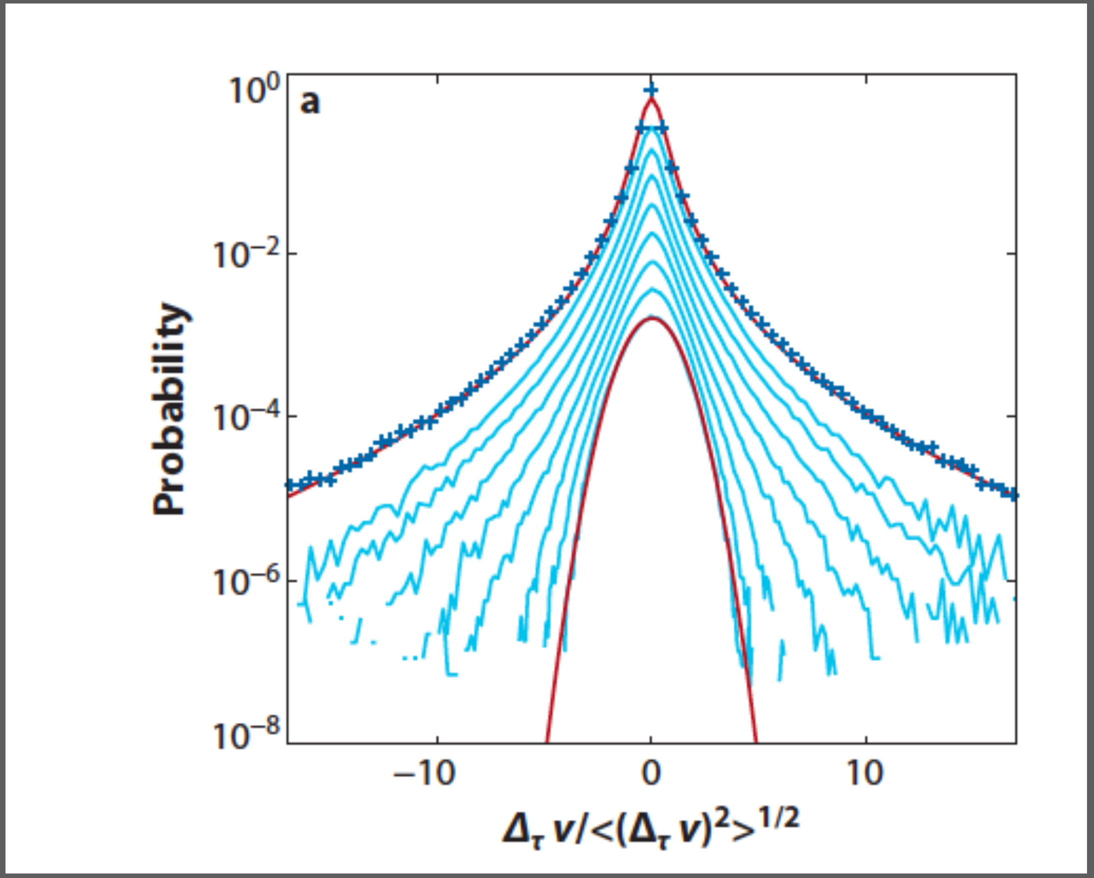
Prediction of Kolmogorov theory for r in the inertial range

$$S_p(r) \equiv \langle (\delta v(r))^p \rangle \sim \epsilon^{p/3} r^{p/3}$$ then we should observe $$\boxed{\frac{S_p(r)}{S_2(r)^{p/2}} \sim const}$$

$$P\left[\frac{\delta v(r)}{S_2(r)^{1/2}}\right]$$ independent of r

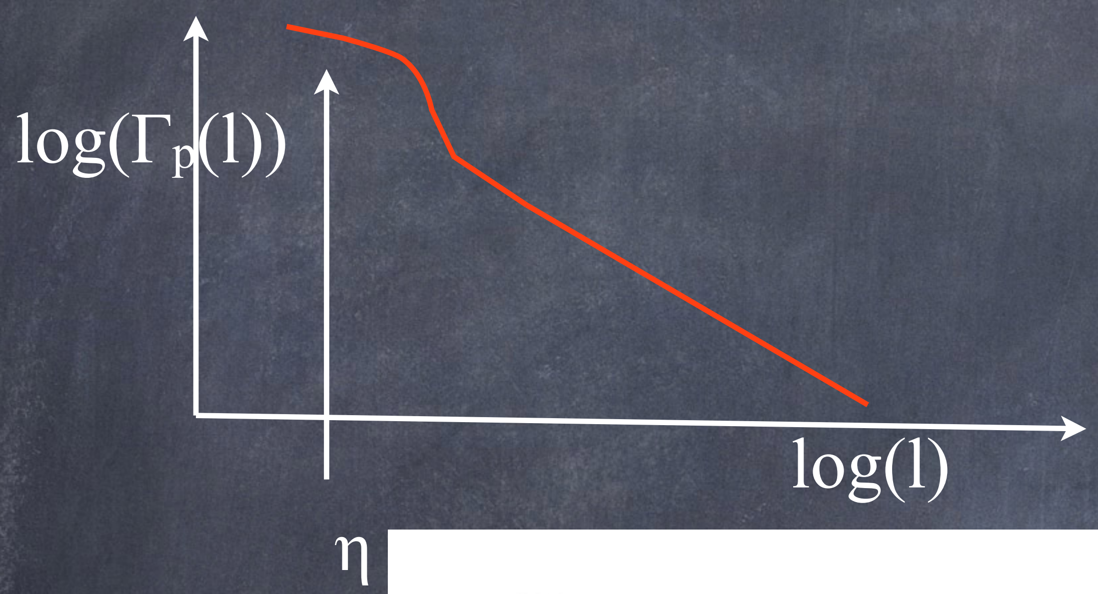
Pdf of δv

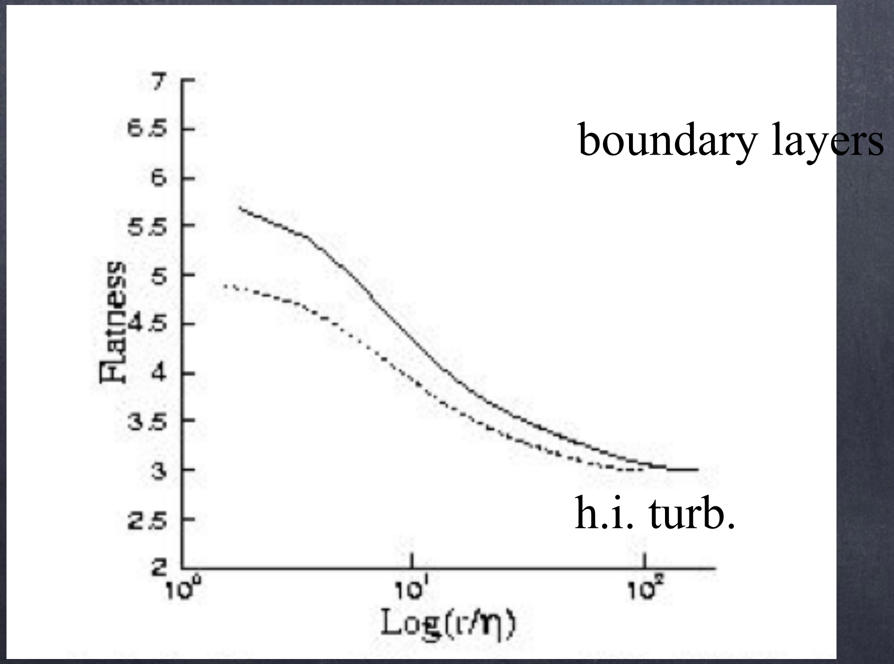


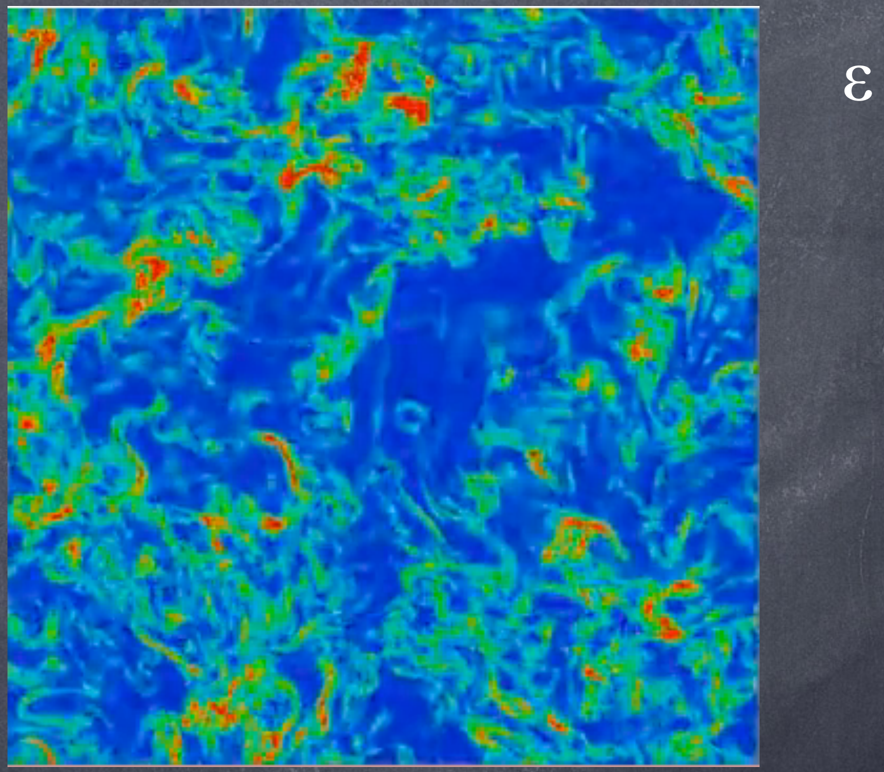


For the Kolmogorov theory to be true, the statistical properties of the inertial range are independent of the forcing mechanism and on the dissipation mechanism (universality). Moreover, we should get

$$S_p(l) = \langle(\delta v(l)^p\rangle \qquad \Gamma_p(l) \equiv \frac{S_p(l)}{S_2(l)^{p/2}} \sim const\ for\ L \ggg \eta$$







strong fluctuations of energy dissipation.

Casciola et al. 2003

Kolmogorov theory is not correct: clear deviations (although small) in the inertial range

What is wrong?

$$\langle (\delta v(r))^p \rangle = \langle \epsilon^{p/3} \rangle r^{p/3}$$

one may wonder whether in computing $\langle \epsilon^{p/3} \rangle$ we need to take into account fluctuations in the energy dissipation

$$\delta v(r) = \epsilon^{1/3} r^{1/3} = U(L) \left(\frac{r}{L} \right)^{1/3} \quad \text{with} \qquad \epsilon \sim \frac{U(L)^3}{L}$$

If U(L) (large scale fluctuations) are gaussian then $\quad P(\epsilon) \sim \frac{1}{\epsilon^{2/3}} \exp\left(\frac{-\epsilon^{2/3}}{2\sigma^2} \right)$

$$\boxed{\text{Energy dissipation can show strong fluctuations. However they are independent of r}}$$

To explain the observations we need some space/time correlation in the energy dissipation and/or in the correlation between $\delta v(r)$ and the energy dissipation.

We need some "statistical/geometrical" features of the system which we are missing.

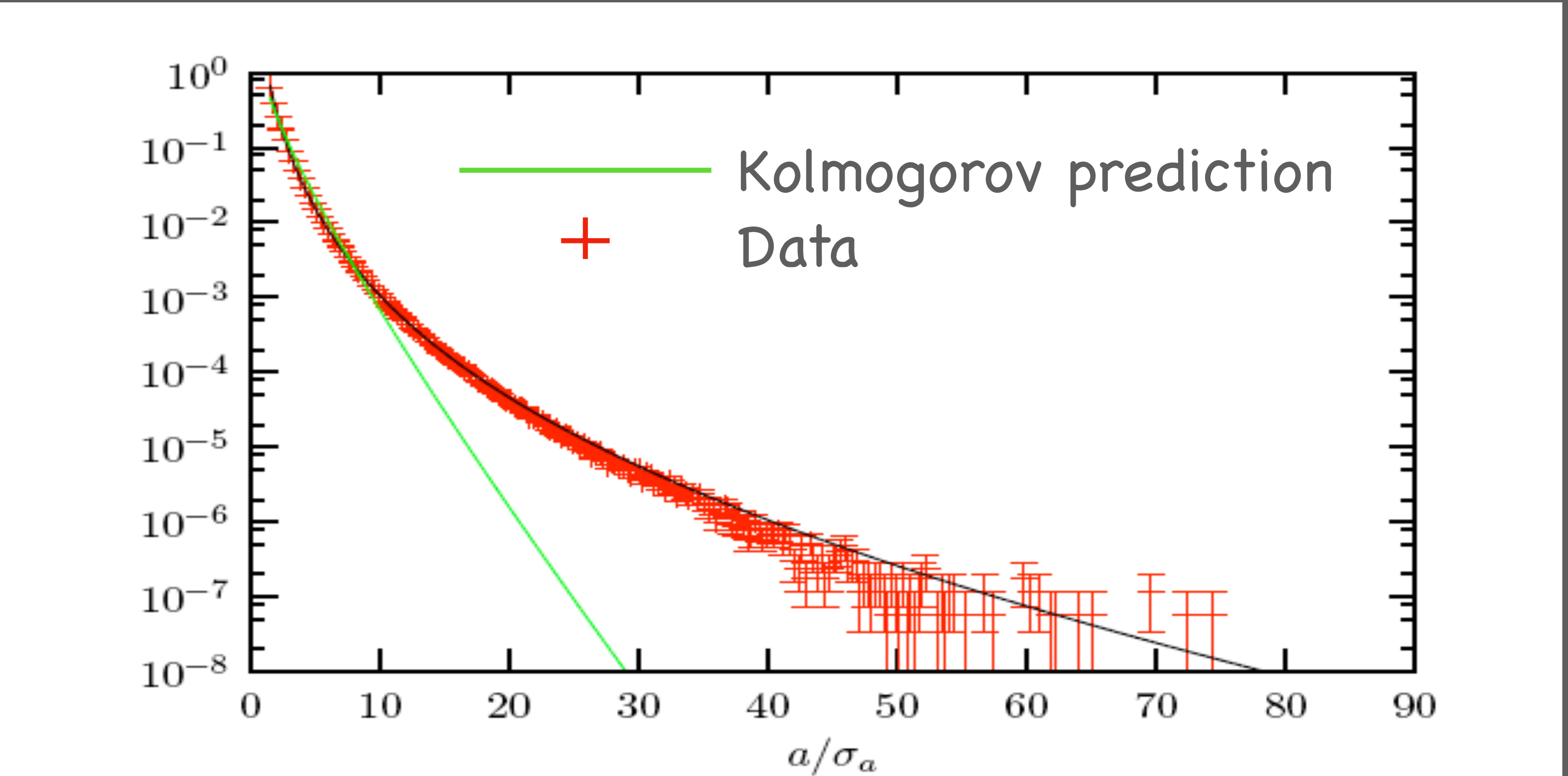
Let us investigate the problem differently. We consider a lagrangian particle which is transported by an homogeneous isotropic turbulent flows at large Re.

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, \mathbf{t})$$

Given **x(t)** we can measure the acceleration a(t). The Kolmogorov theory suggests

$$a \sim \frac{\delta v(\eta)}{\tau_\eta} \sim Re^{1/4}\epsilon^{3/4}$$

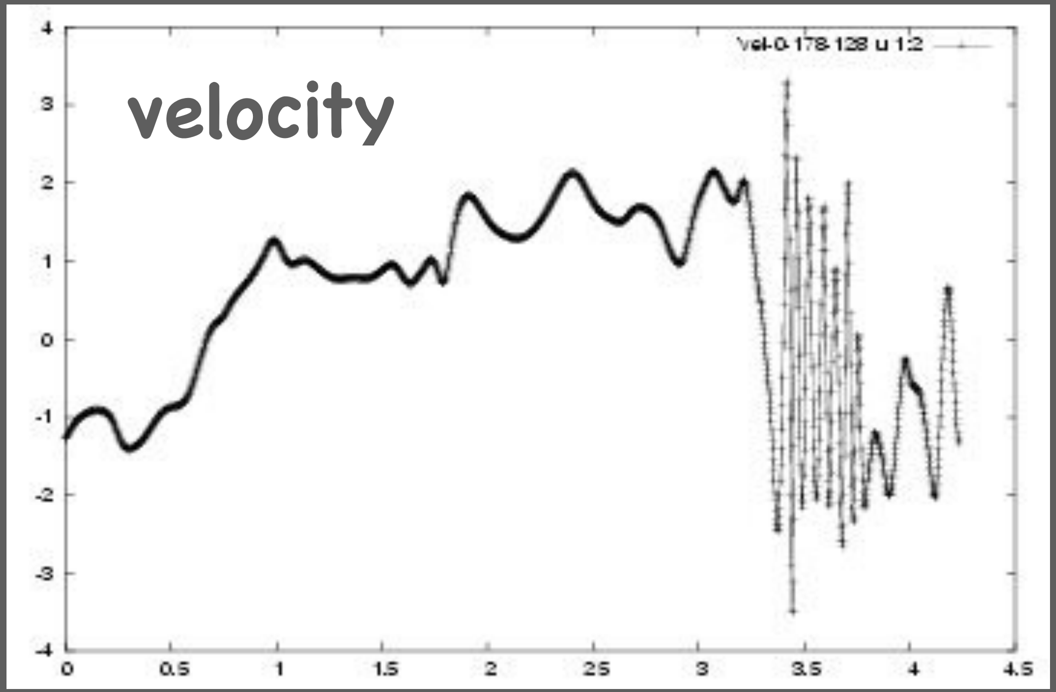
Knowing P[ε] (in the Kolmogorov theory) we can compute P[a]



L Biferale, G Boffetta, A Celani, BJ Devenish, A Lanotte, F Toschi 2005

Dissipation are not properly taken into account by the Kolmogorov theory

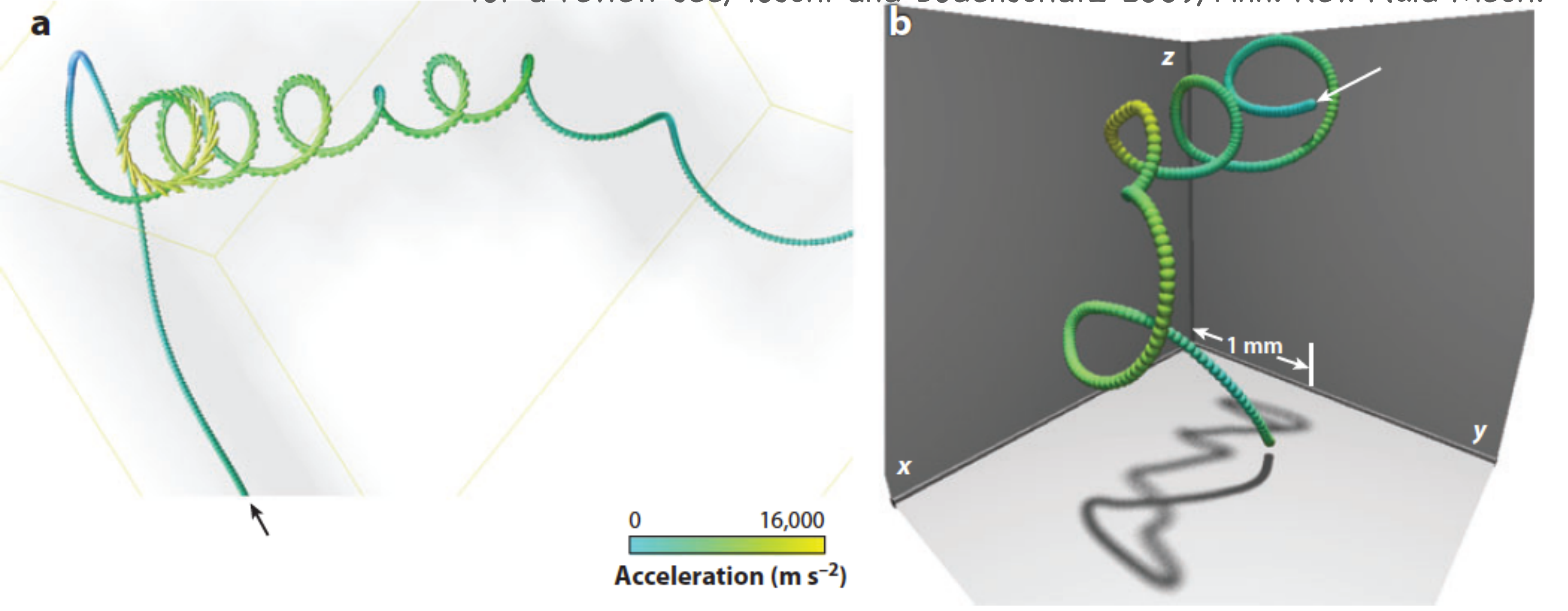
Biferale, Boffetta, Celani, Lanotte,Toschi, 2005



A closer look to the acceleration

The acceleration shows strong fluctuations when the particle becomes trapped in a vortex filaments

for a review see, Toschi and Bodenschatz 2009, Ann. Rev. Fluid Mech.



La Porta, A., Voth, G.A., Crawford, A.M., Alexander, J. & Bodenschatz, E. 2001l Nature 2001

The previous result suggests different scenarios

- small scale turbulent fluctuations are dominated by vortex tubes or coherent structures immersed in a "random sea" of Kolmogorov-like fluctuations; (implications: non universality of inertial range properties; non trivial dependence on the dissipation mechanism;….)

- for extremely large Re, all deviations with respect the Kolmogorov theory disappear

- we still assume that "scaling" of $\delta v(r)$ versus r is true and we must look for a generalisation of the Kolmogorov theory (implications: universality in the scaling of $S_p(r)$ ….)

Some remarks on the scale invariance of the Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} \qquad r \to \lambda r \qquad \begin{aligned} v &\to \lambda^h v \\ t &\to \lambda^{1-h} t \\ \nu &\to \lambda^{1+h}\nu \end{aligned}$$

$$\epsilon \to \lambda^{3h-1}\epsilon$$

h=1/3 is the value selected by the Kolmogorov theory

We generalise the scale invariance by assuming that any h is possible although for each h we give a "weight" proportional to $\lambda^{3-D(h)}$

Parisi Frish 1983, RB, Paladin, Parisi, Vulpiani 1984

This generalisation implies that the energy dissipation is a multifractal field.

D(h) is can be thought to be the fractal dimension of the set for which δv(r) ~ r^h

$\delta v(r) \sim r^h \qquad P(h) \sim r^{3-D(h)}$

$$S_p(r) = \int dh r^{ph} r^{3-D(h)} \sim r^{\zeta(p)}$$

$$\zeta(p) = inf_h[ph + 3 - D(h)]$$

$\zeta(p) \leq \frac{p}{2}\zeta(2)$ **(anomalous) scaling)**

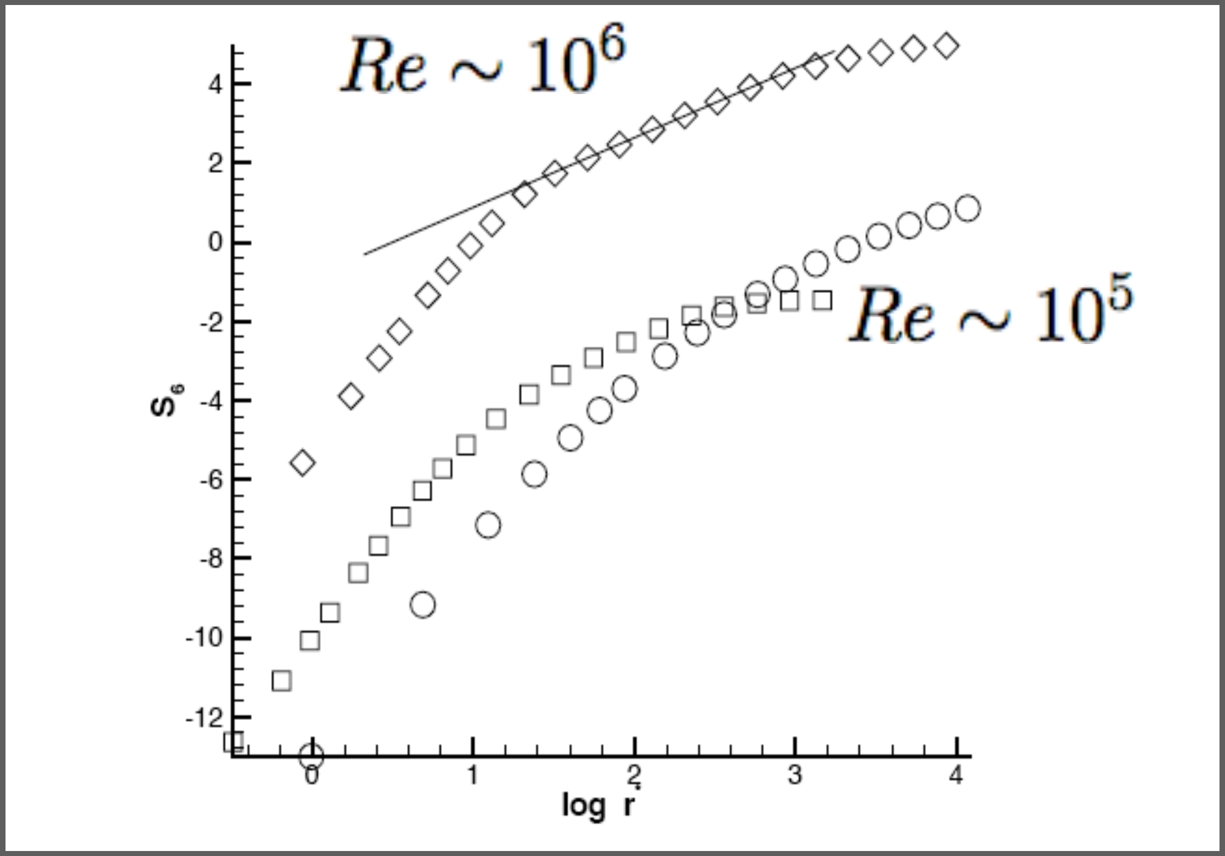
$\zeta(3) = 1$ Kolmogorov exact result

If the multifractal generalisation is the correct way to describe turbulent fluctuations we need to answer several questions:

- is that true that $S_p(r) \sim r^{\varsigma(p)}$ with $\varsigma(p)$ non linear function of p? (anomalous scaling)
- is the anomalous scaling universal ?
- how anomalous scaling can explains strong fluctuations near the dissipation range (i.e. acceleration)?
- how can we compute D(h) and/or $\varsigma(p)$ from the equation of motions?

Knowing $\varsigma(p)$ we can compute D(h) by the inverse Legendre transform.

test of anomalous scaling in turbulent flows

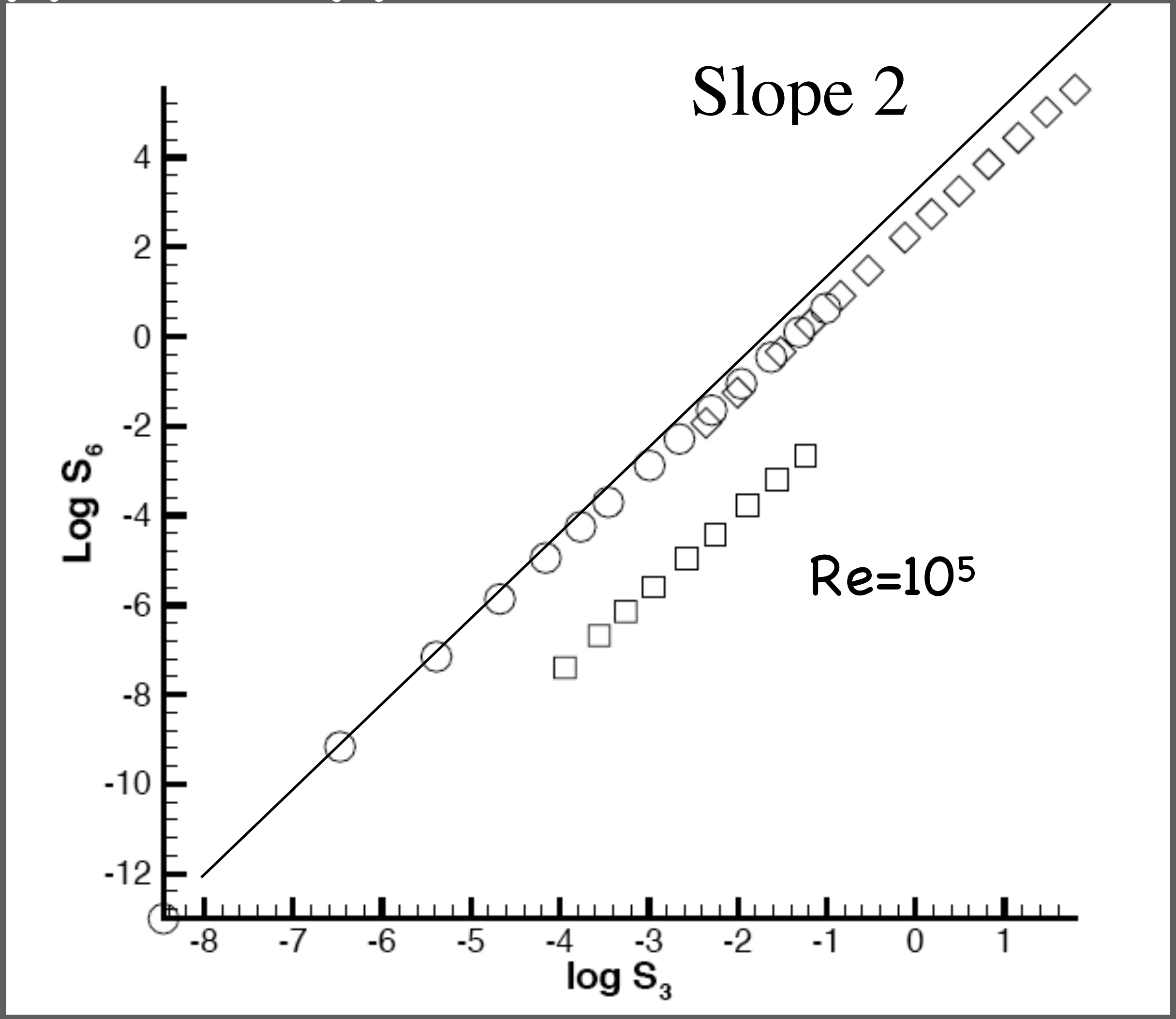


The value of ς(p) depends how to make the fit and we do not know how to compute finite size effects in Re

The meaning of anomalous scaling

A : random variables

$\langle A(r)^p \rangle \neq \langle A(r) \rangle^p$

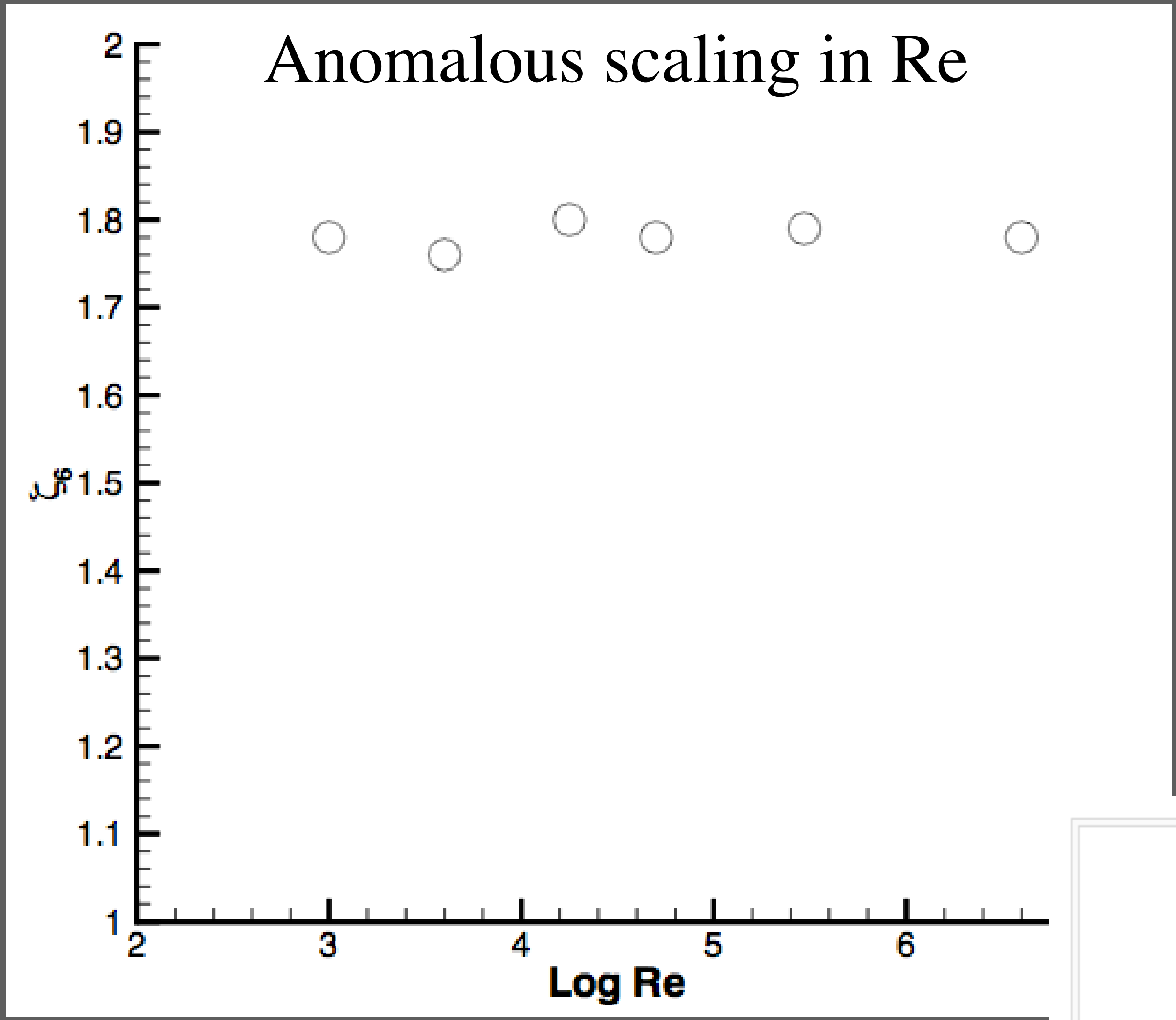


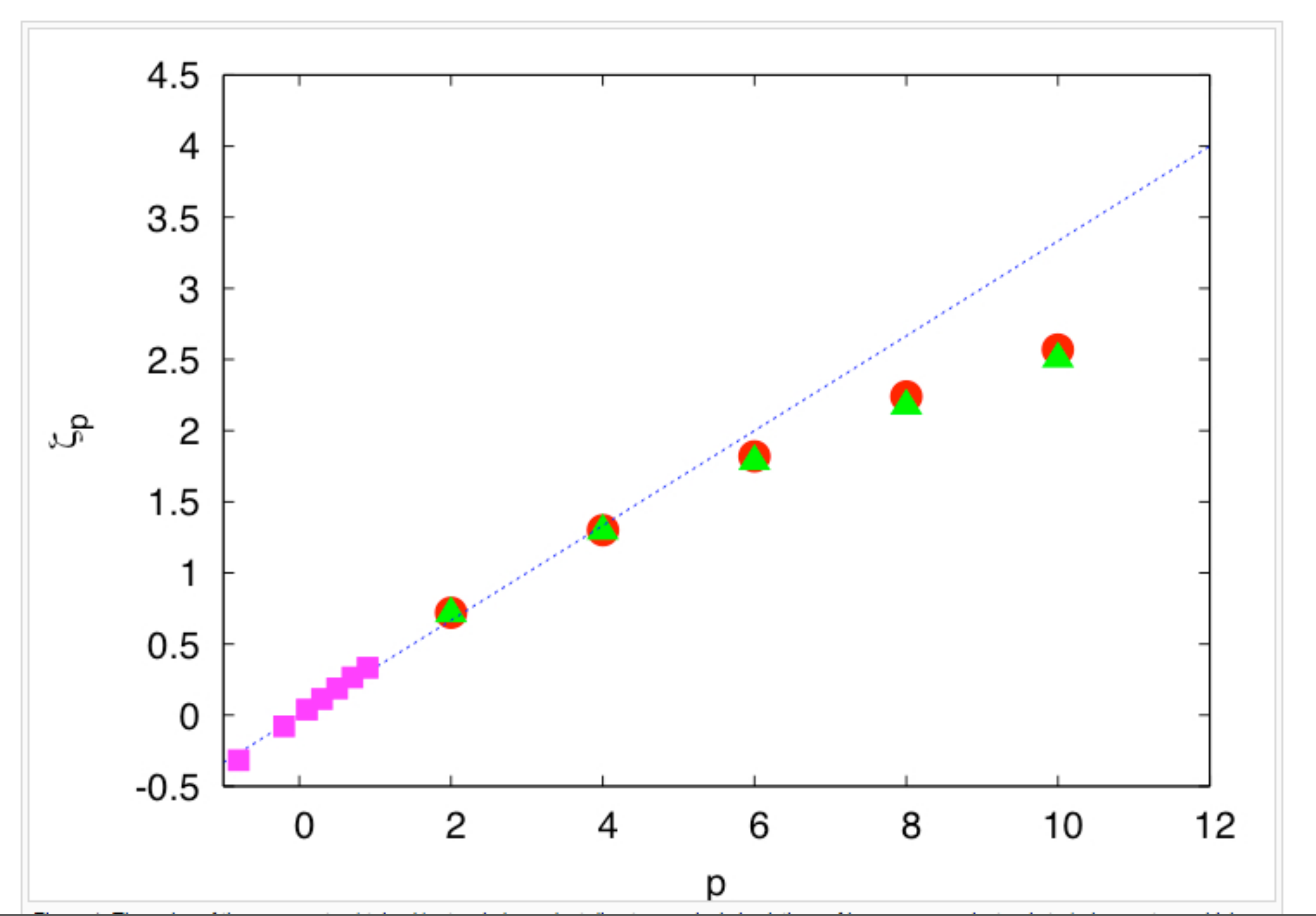
K41 $S_p(r) = S_3(r)^{p/3}$

Extended self similarity

R,B, Baudet Ciliberto Massaioli
Tripiccione Succi 1993

RB, Biferale,Ciliberto, et. a





Courtesy by L. Biferale

SHELL MODELS: simplified model of turbulence to study anomalous scaling and multifractal behavior (for a review see Biferale 2006)

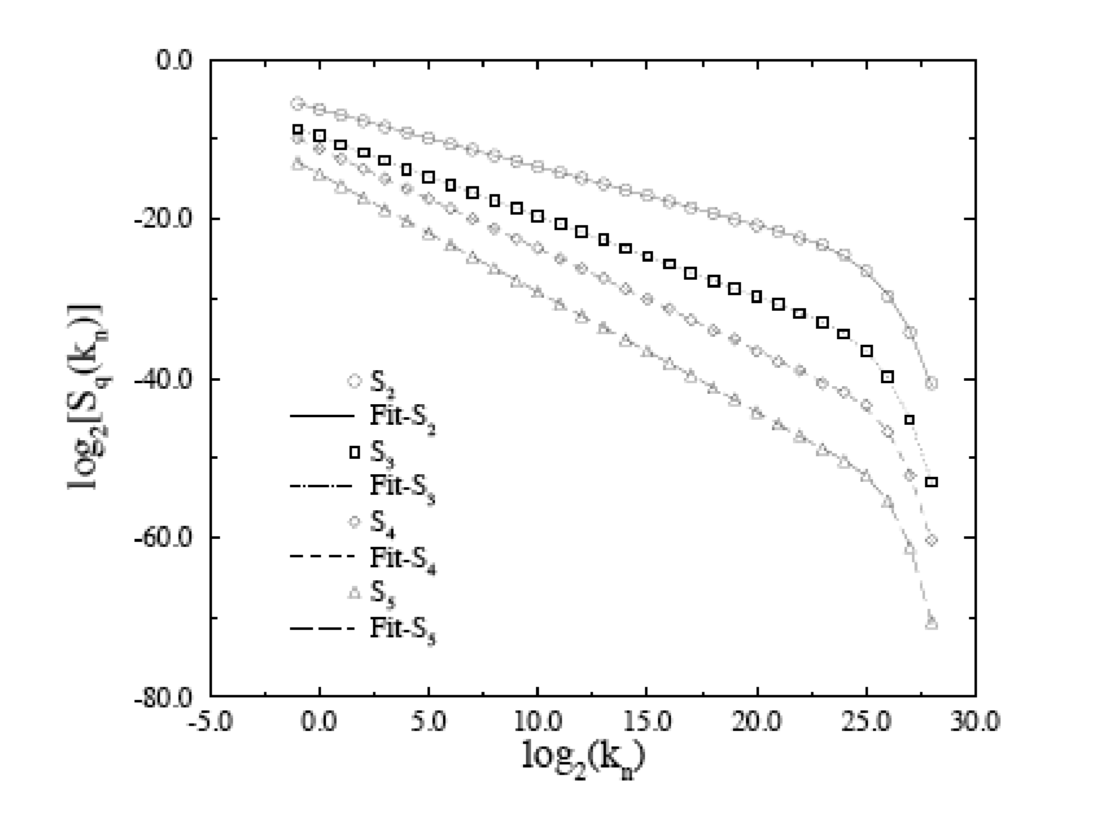
$k_n = 2^n k_0 \longrightarrow u_n$ complex variable
non linear interactions with required invariants and scale symmetry

$$\frac{du_n}{dt} = i\left(ak_{n+1}u_{n+2}u^*_{n+1} + bk_n u_{n+1}u^*_{n-1} - ck_{n-1}u_{n-1}u_{n-2}\right) - \nu k_n^2 u_n + f_n\,,$$

a=1, b= -0.4, c= -0.6

AMAZING !!!

q	S_q	$\langle \lvert u_n \rvert^q \rangle$
1		0.393 ± 0.006
2	0.720 ± 0.008	0.720 ± 0.008
3	1.000 ± 0.005	1.003 ± 0.009
4	1.256 ± 0.012	1.256 ± 0.012
5	1.479 ± 0.006	1.488 ± 0.013
6	1.706 ± 0.015	1.706 ± 0.015
7	1.901 ± 0.010	1.910 ± 0.020



Numerical simulations of shell models support another view of multifractal field

Superposition of "istanton like" solution: local scaling

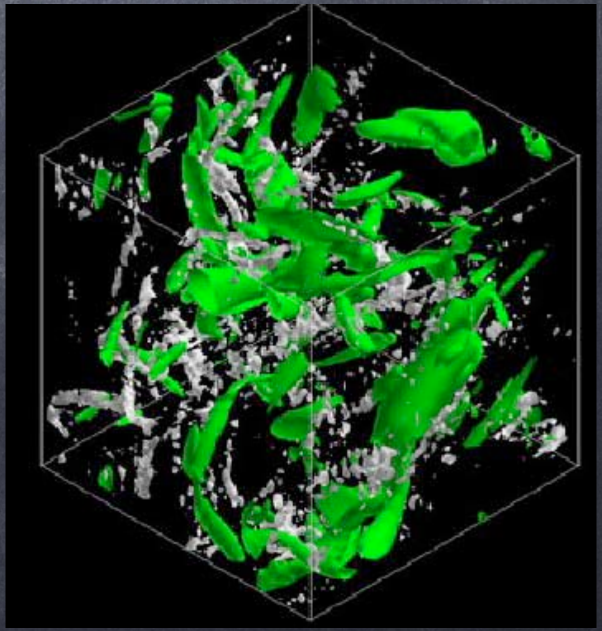
$$\tilde{v}(k,t) = \Sigma k^{-h} f_h[(t-t_*)k^{1+h}]$$

Siggia 78, Parisi 92, Daumont, Dombre, Gibson 99, Biferale, Daumont, Dombre, Lanotte 99L'vov 2002, ...

With suitable probability distribution for "h" and "t_*"

The asymptotic limit for $p\to\infty$: link with coherent structures?

$$\zeta_p \to ph_0 + 3 - D_0 \xrightarrow{\quad ? \quad}$$



IS MULTIFRACTALITY RELATED TO STRUCTURES ?

The fluctuations near the dissipative scale can be computed assuming anomalous scaling in the inertial range.

Let us apply this argument using the multifractal framework.

$$\boxed{\frac{\delta v(\eta)\eta}{\nu} \sim 1}$$ definition (scaling wise) of the dissipation scale

$$\delta v(\eta) = U_0 \left(\frac{\eta}{L}\right)^h \qquad P[h] = \left(\frac{\eta}{L}\right)^{3-D(h)}$$ U_0 assumed gaussian

$$\boxed{\eta = \eta(h) = Re^{-\frac{1}{1+h}} \qquad P[h] = Re^{-\frac{3-D(h)}{1+h}}}$$ the dissipation scale fluctuates!

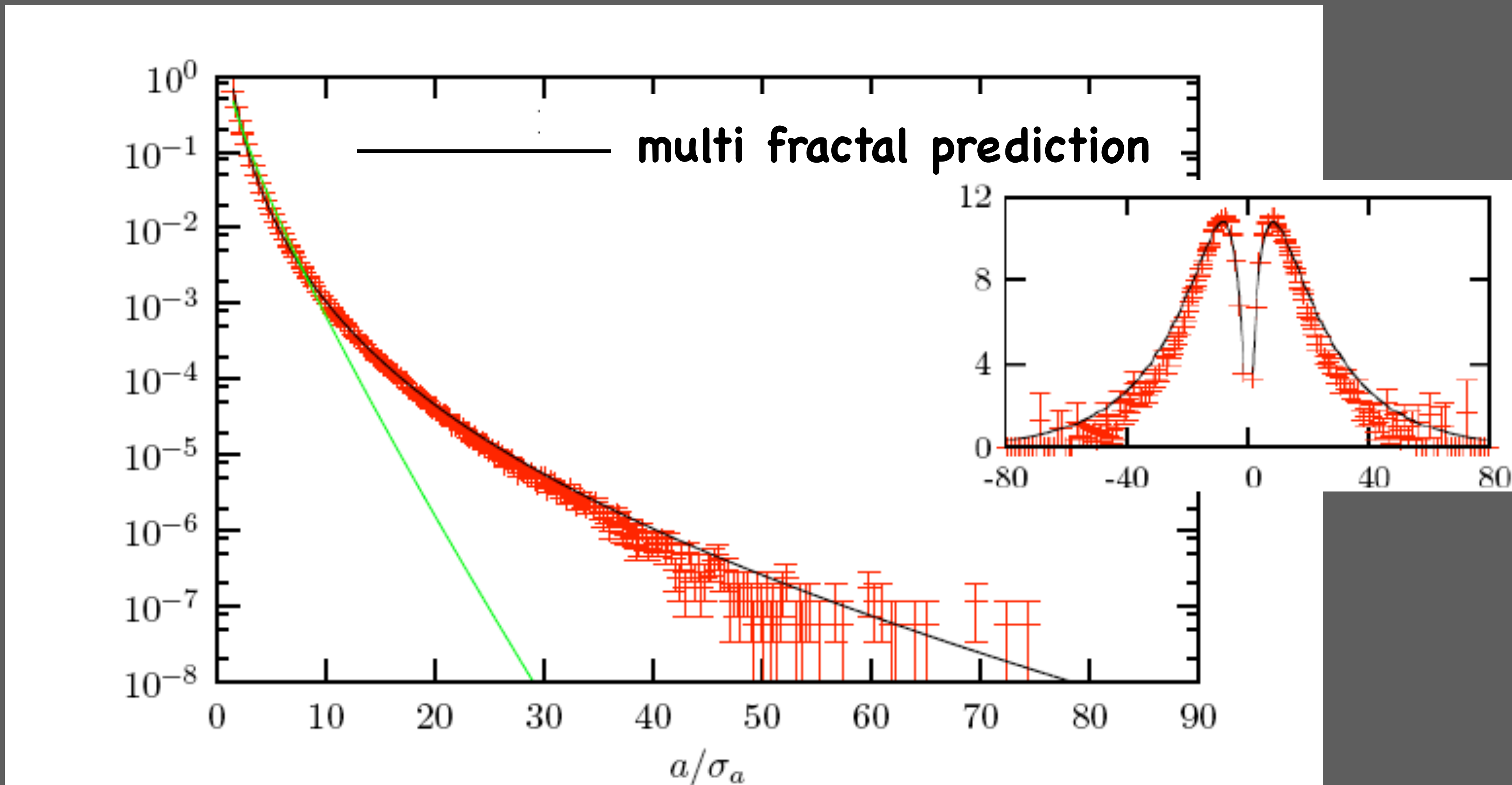
Frisch, U.& Vergassola, M. 1991.
Paladin,G. & Vulpiani, A. 1987

$$\langle (\nabla v)^{2p} \rangle = \langle \left(\frac{\delta v(\eta)}{\eta}\right)^{2p} \sim Re^{\chi(p)} \qquad \chi(p) = sup_h \left[\frac{2p(1-h) - 3 + D(h)}{1+h}\right]$$

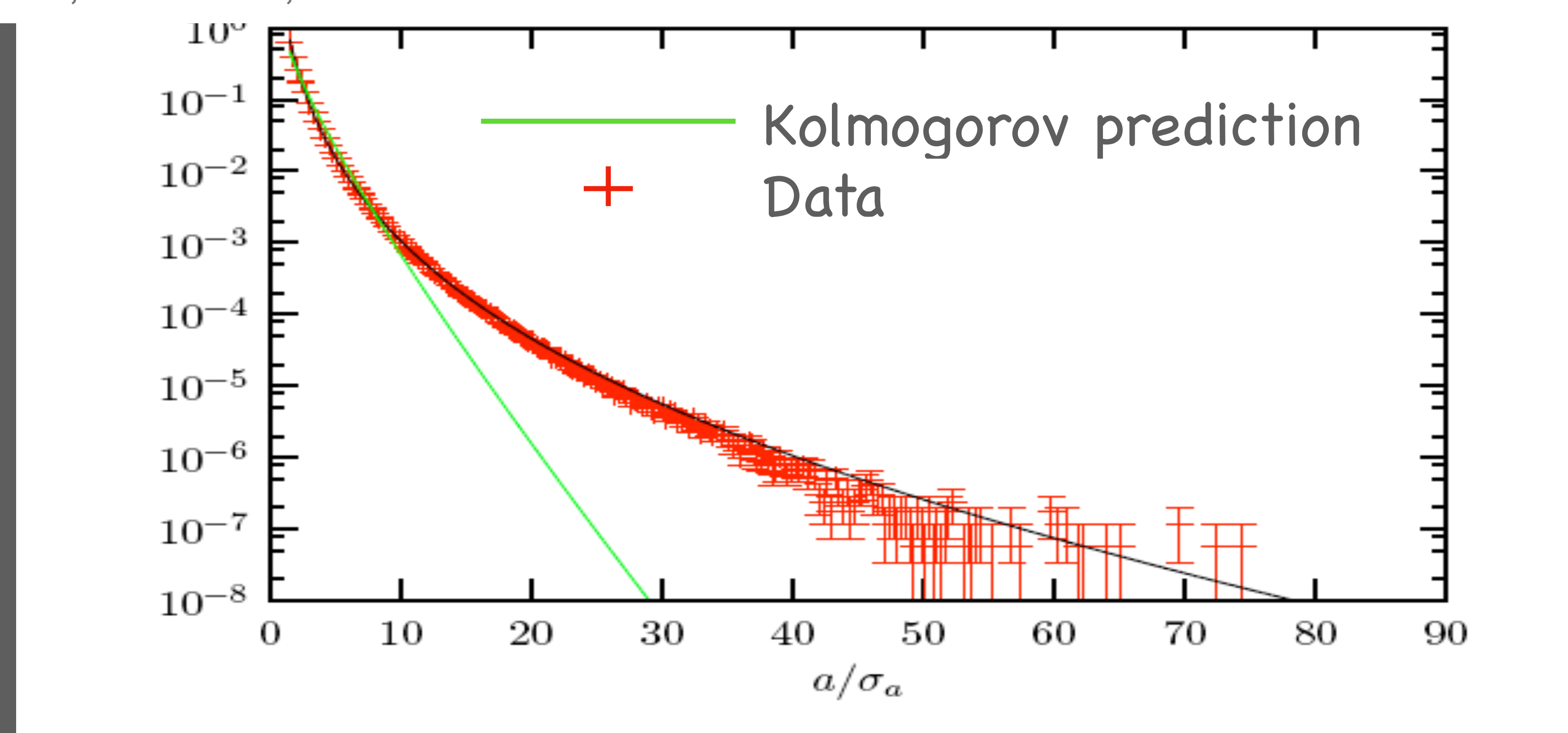
We can compute the probability distribution of the acceleration a as $a = \dfrac{\delta v^2(\eta)}{\eta}$

Remark: we estimate D(h) using inertial range anomalous scaling.

Non trivial prediction based on the multifractal theory.



L Biferale, G Boffetta, A Celani, BJ Devenish, A Lanotte, F Toschi 2005



Dissipation range in for Lagrangian Turbulence

The point of view of a lagrangian particle.

$$\frac{\eta}{L} \sim \frac{\nu^{3/4}}{\epsilon^{1/4} L} \sim L Re^{-3/4}$$

$$\tau_\eta = \frac{\eta}{\delta v(\eta)} \qquad \frac{\eta \delta v(\eta)}{\nu} = 1$$

$$\frac{\tau_\eta}{T} = \frac{\tau_\eta}{L/U} \sim T Re^{-1/2}$$

Lagrangian dynamics enables us to study in a better way the effect of dissipation

Lagrangian dynamics means to study the velocity field experienced by a particle driven by the turbulent flow.

Eulerian structure functions

$$S_p(r) = \langle (\delta u(r))^p \rangle \sim r^{\zeta_E(p)}$$

Lagrangian structure functions

$$S_p(\tau) = \langle (v(t+\tau) - v(t))^p \rangle \sim \tau^{\zeta_L(p)}$$

$$\delta v(\tau) \sim \delta u(r) \qquad \tau \sim \frac{r}{\delta u(r)}$$

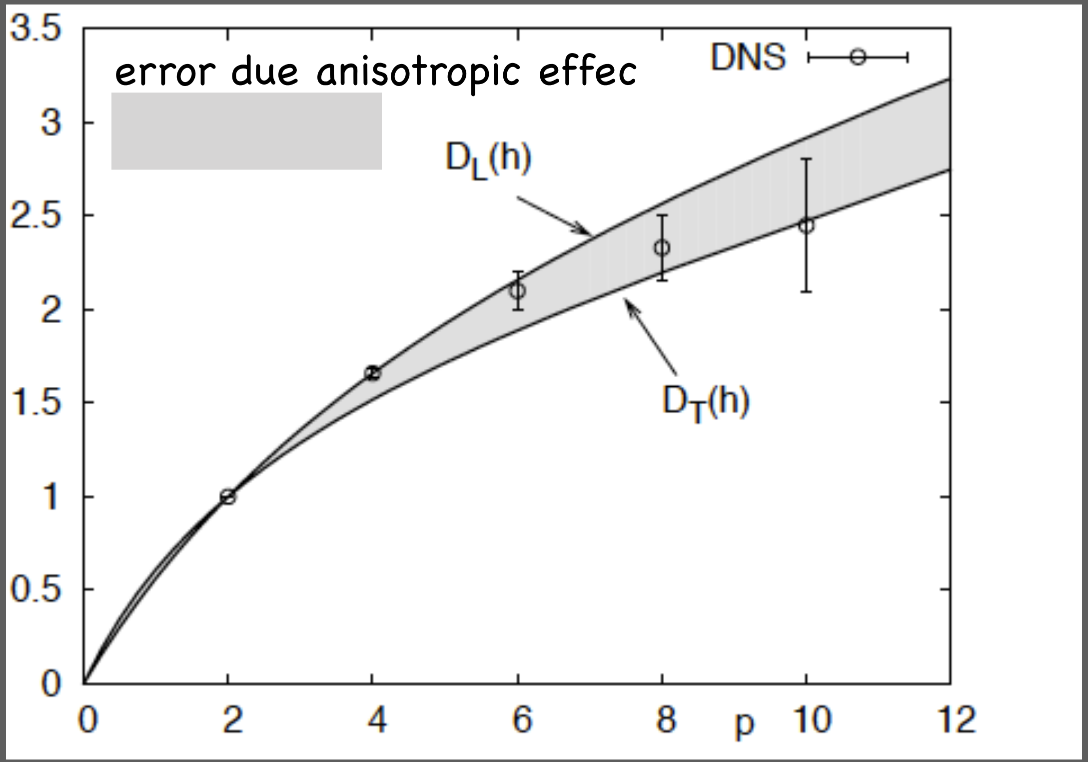
$$r \to \lambda r \qquad u \to \lambda^h u \qquad \tau \to \lambda^{1-h} \tau$$

Lagrangian exponents can be predicted from the Eulerian exponents
Borgas, MS. 1993

$$\zeta_E(p) = inf_h \left[ph + 3 - D(h) \right]$$

$$\zeta_E(p) \to \zeta_L(p) = inf_h \left[\frac{ph + 3 - D(h)}{1-h} \right]$$

R B, L Biferale, R Fisher, DQ Lamb, F Toschi 2010

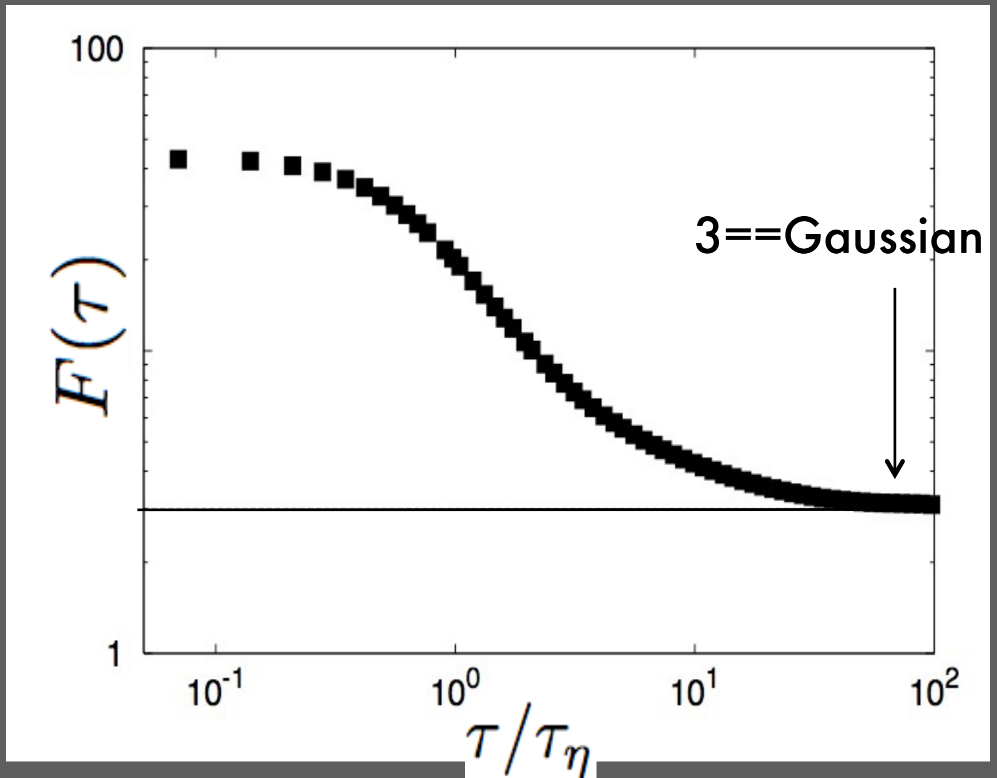


Intermittency and coherent structures

$$S_4(\tau) = S_2(\tau)^{a(\tau)}$$

$$F(\tau) = S_2(\tau)^{a(\tau)-2}$$

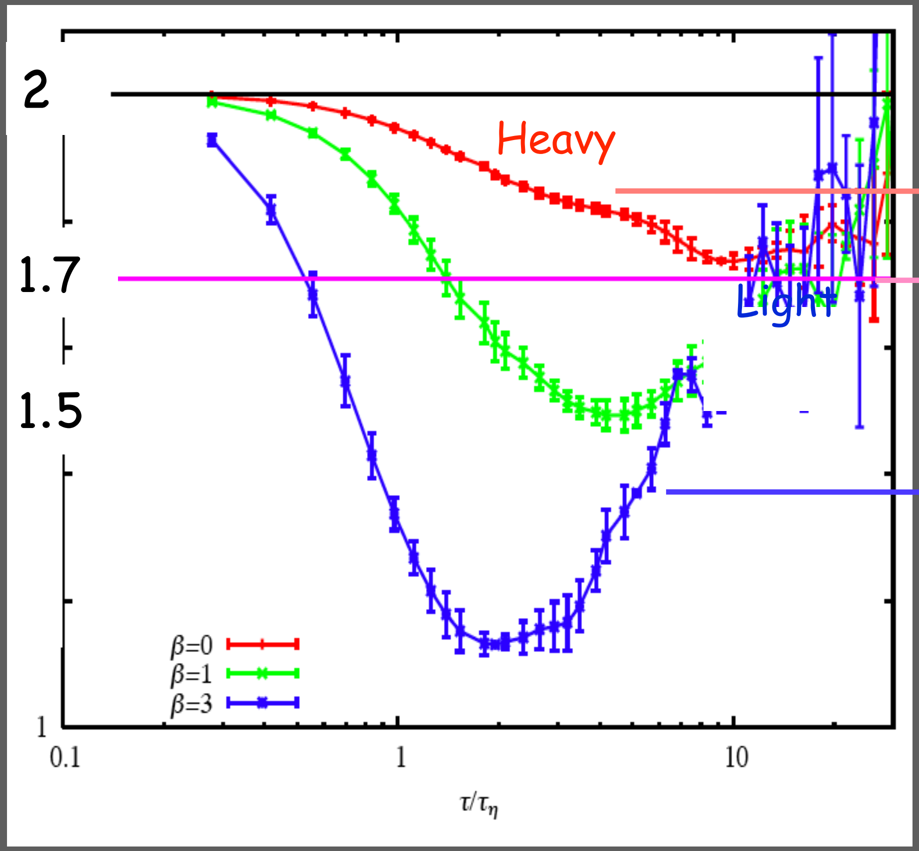
$$F(\tau) = \frac{S_4(\tau)}{S_2(\tau)^2}$$



$$a(\tau) = \frac{dlogS_4(\tau)}{dlogS_2(\tau)}$$

$a(\tau)$ can be considered as a local measure (in τ) of intermittency

plot from E. Calzavarini



Effects of vortices on intermittency behavior

Outside vortices

anomalous scaling in the inertial range

trace

Inside vortices

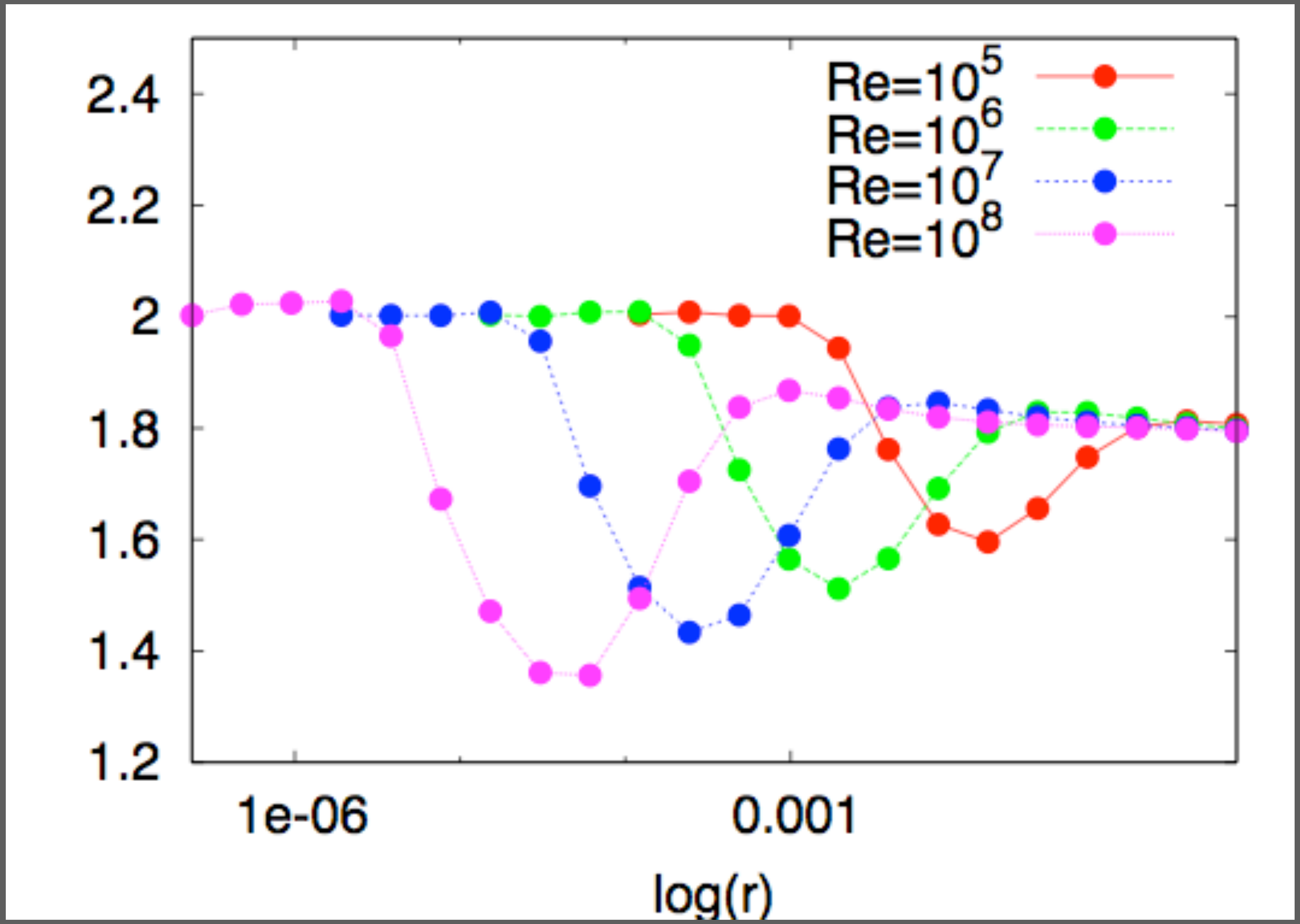
J Bec, L Biferale, G Boffetta, M Cencini, S Musacchio, F Toschi 2006

R Benzi, L Biferale, E Calzavarini, D Lohse, F Toschi 2009

L Biferale, G Boffetta, A Celani, A Lanotte, F Toschi 2005

Theoretical prediction of the MF for different Reynolds: intermittency shows up before scaling.

$$a(r) = \frac{dlogS_4(r)}{dlogS_2(r)}$$

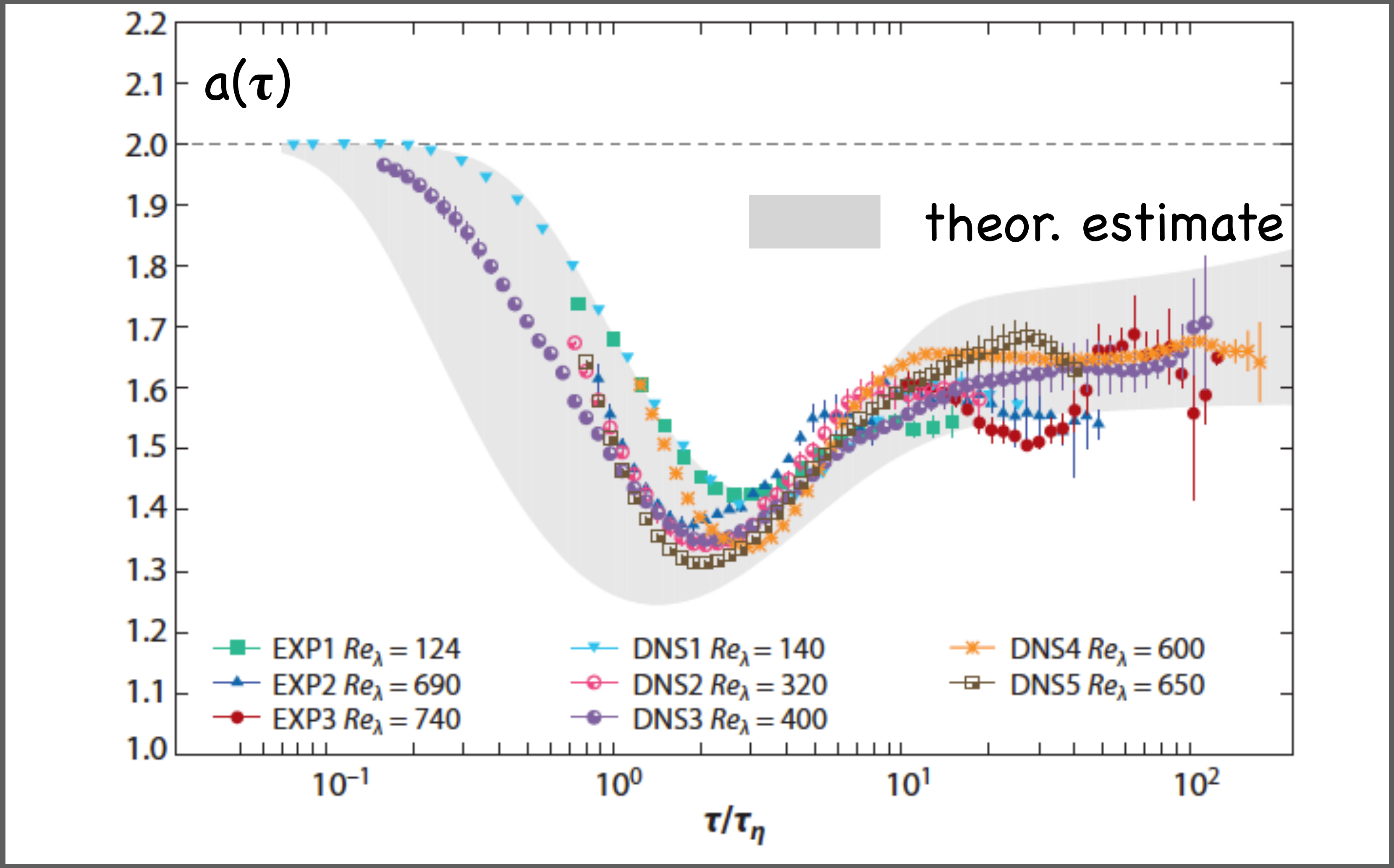


Benzi, Biferale 2010

The inertial properties of turbulence (i.e. the function D(h)) are able to explain quantitatively the increase of intermittency observed in the dissipation range.

Vortices appear to be relevant in the dissipation range, although their statistical properties are constrained by the inertial range dynamics and do not affect anomalous scaling.

$$a(\tau) = \frac{dlogS_4(\tau)}{dlogS_2(\tau)}$$



Major international effort, Arneo

What did we obtain?

Open questions:

- we still cannot "prove" the existence of viscous anomaly in the Navier-Stokes equation, although we have a deep physical intuition on why it occurs (breaking of lagrangian trajectories);

- we are not (yet) able to compute D(h) and/or the anomalous exponents (maybe easier than the problem of viscous anomaly);

What we get (A LOT!)

1. we have a consistent way to describe turbulence both in the inertial range and in the dissipative range (independent on the existence of vortices or other coherent structures);

2. the multifractal function D(h) is able to **predicts** (so far) all the scaling properties in r or Re for quantities which are invariant under the same group of transformation of the Navier-Stokes equations

3. we know how to estimate finite size effects in Re and we have good evidence that D(h) is universal;

4. the statistical properties of turbulent fluctuations are independent on the large scale forcing and on the detailed mechanism of energy dissipation (universality);

5. some of the above results can be generalised for non isotropic forcing;

How can we use our results?

- improvement of numerical schemes for realistic applications (machine learning approach?);

- basic understanding of turbulence effects in many scientific problems:
  - statistics of non inertial particles;
  - turbulence effects in population dynamics and evolution (phytoplankton) in the ocean;

- we can explore similarities and/or differences (if any) for fluid turbulence with different physical properties (incomplete list):
  - superfluid turbulence;
  - magnetohydrodynamics;
  - complex fluids (for instance two phase flows, fluids with polymer...)
  - geophysical flows (turbulence in stratified medium, rotating turbulence...);
  - looking at the above problems we may eventually understand how to compute D(h);

Focus of the second talk

We consider three different problems:

1) large scale reversal in turbulent flows and in particular magnetic reversal in dynamo

2) how we may obtain something non trivial using new development in machine learning

3) how turbulence may be used to understand some basic properties of avalanche dynamics

The case of magnetic reversals in the dynamo

$$\partial_t \vec{u} + \vec{u} \bullet \nabla \vec{u} = \vec{B} \bullet \nabla \vec{B} - \frac{\nabla p}{\rho} + \nu \Delta \vec{u} \quad \textbf{+forcing}$$

$$\partial_t \vec{B} + \vec{u} \bullet \nabla \vec{B} = \vec{B} \bullet \nabla \vec{u} + \nu_m \Delta \vec{B}$$

Experimental results

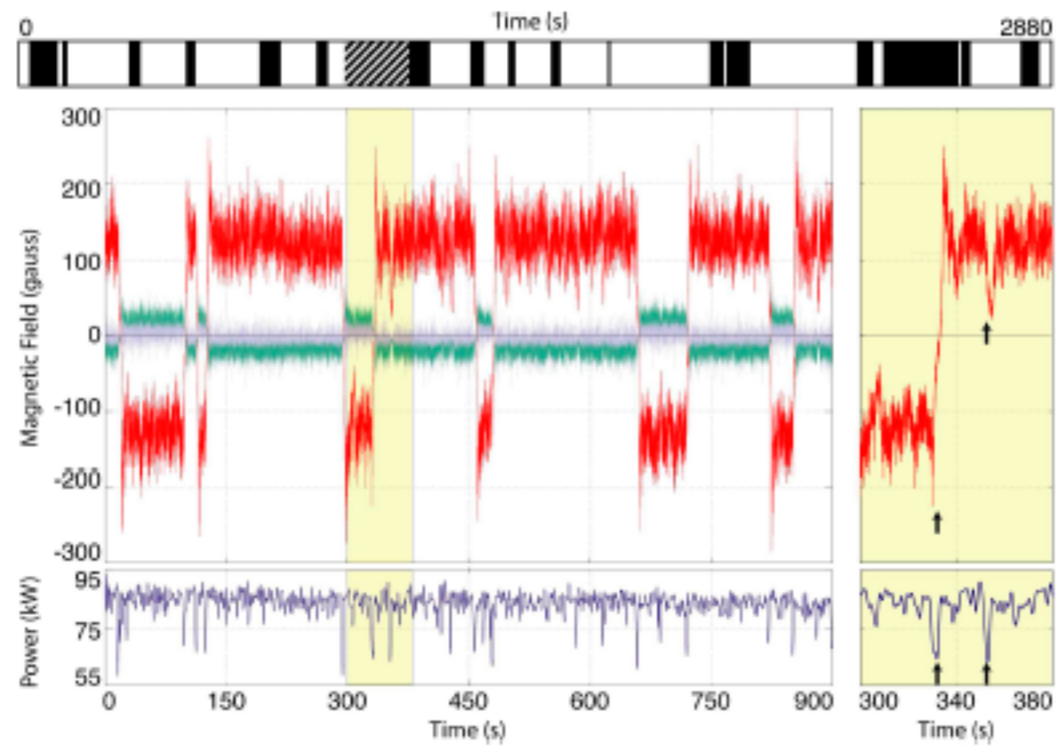


Fig. 3: Magnetic field measured inside the flow vessel, by a 3-dimensional Hall probe. No external magnetic field is applied, other than the ambient field, whose amplitude is about 0.2 gauss across the measurement volume.The temperature of the outer copper cylinder is $T = 123$ °C. (Main): Time evolution of all three magnetic field components. The main component (red) is the azimuthal one. Note that all components decay to zero at a reversal. The bottom graph shows synchronous recordings of the power driving the flow. (Right): detail of the time series of the main magnetic field and simultaneous power consumption (arrows mark the synchronous events). (Top): Chronos of the magnetic field orientation, white for a positive direction, black for the negative direction, for 2 successive recordings 900 and 1800 seconds long (separated by the shaded area, the first sequence corresponds to the main graph). In this regime, the von Kármán flow is driven with counter-rotating disks at frequencies $F_1 = 16$ Hz and $F_2 = 22$ Hz.

Large scale mechanism or "turbulent noise"?

What can we learn using a simple approach?

There are two important numbers in the systems $Re = UL/\nu$ and $Re_m = UL/\nu_m$. Geophysical dynamo are believed to occurs for $Re_m \ll Re$ and for $Re \gg 1$: very (!) difficult problem for direct numerical simulations.

To make progress we (RB, JF. Pinton 2009) use a simplified approach:
the vector fields u and B are replaced by "shell variables" U_n and B_n
$k_n = 2^n$, n=1..25. U_n and B_n are complex numbers which satisfy the equations:

$$\frac{du_n}{dt} = \frac{i}{3}(\Phi_n(u,u) - \Phi_n(B,B)) - \nu k_n^2 u_n + f_n \,,$$
$$\frac{dB_n}{dt} = \frac{i}{3}(\Phi_n(u,B) - \Phi_n(B,u)) - \nu_m k_n^2 B_n \,,$$

$$\frac{i}{3}\Phi_n(u,B) \to -\mathbf{u} \bullet \nabla \mathbf{B}$$

where Φ_n is a non linear operator which (for $\nu=\nu_m=0$) satisfies the requirements of energy and cross-helicity conservations.

$$\begin{aligned}\Phi_n(u,w) &= k_{n+1}[(1+\delta)u_{n+2}w^*_{n+1} + (2-\delta)u^*_{n+1}w_{n+2}] \\ &+k_n[(1-2\delta)u^*_{n-1}w_{n+1} - (1+\delta)u_{n+1}w^*_{n-1}] \\ &+k_{n-1}[(2-\delta)u_{n-1}w_{n-2} + (1-2\delta)u_{n-2}w_{n-1}] \,, \qquad (3)\end{aligned}$$

We assume a saturation in the large scale magnetic field and $B_1=0$ (geometry constrain on the largest scale)

$$\frac{dB_2}{dt} = \frac{i}{3}[\Phi_2(u,B) - \Phi_2(B,u)] - a_m B_2^3 - \nu_m k_2^2 B_2$$

$$F_2(u,B) \equiv \frac{i}{3}[\Phi_2(u,B) - \Phi_2(B,u)]$$

This quantity represents the non linear interaction due to larger ($n<2$) and smaller ($n>2$) scales. Also we expect, for small enough ν_m, that dynamo occurs. Thus we expect that "on the average" $F_2(u,B)$ represents an instability for the amplitude of the shell B_2

$$F_2(u,B) = \beta B_2 + f(t)$$

where $\beta>0$ is the instability and $f(t)$ are fluctuations (eventually strong)

The behaviour of B_2 can be argued to satisfies approximately the equation

$$\frac{dB_2}{dt} = \beta B_2 - a_m B_2^3 - \nu_m k_2^2 B_2 + f(t)$$

We now understand the effect of the term $a_m B_2^3$

This term breaks the rotational invariant in the complex plane $B_2 = B_{2r} + B_{2i}$

Rotational invariance is preserved by the non linear interactions (the term proportional to β).

We are assuming that the large scale magnetic field B_2 reaches a saturation for $B_{2i} \sim 0$ and finite value of B_{2r} (positive or negative).

This conclusion can be reached by looking at the equation for $|B_2|^2$

$$\frac{d|B_2|^2}{dt} = \beta|B_2|^2 - a_m|B_2|^2(B_{2r}^2 - B_{2i}^2) - \nu_m k_2^2|B_2|^2 + ...$$

Neglecting the …… terms, equilibrium is reached if

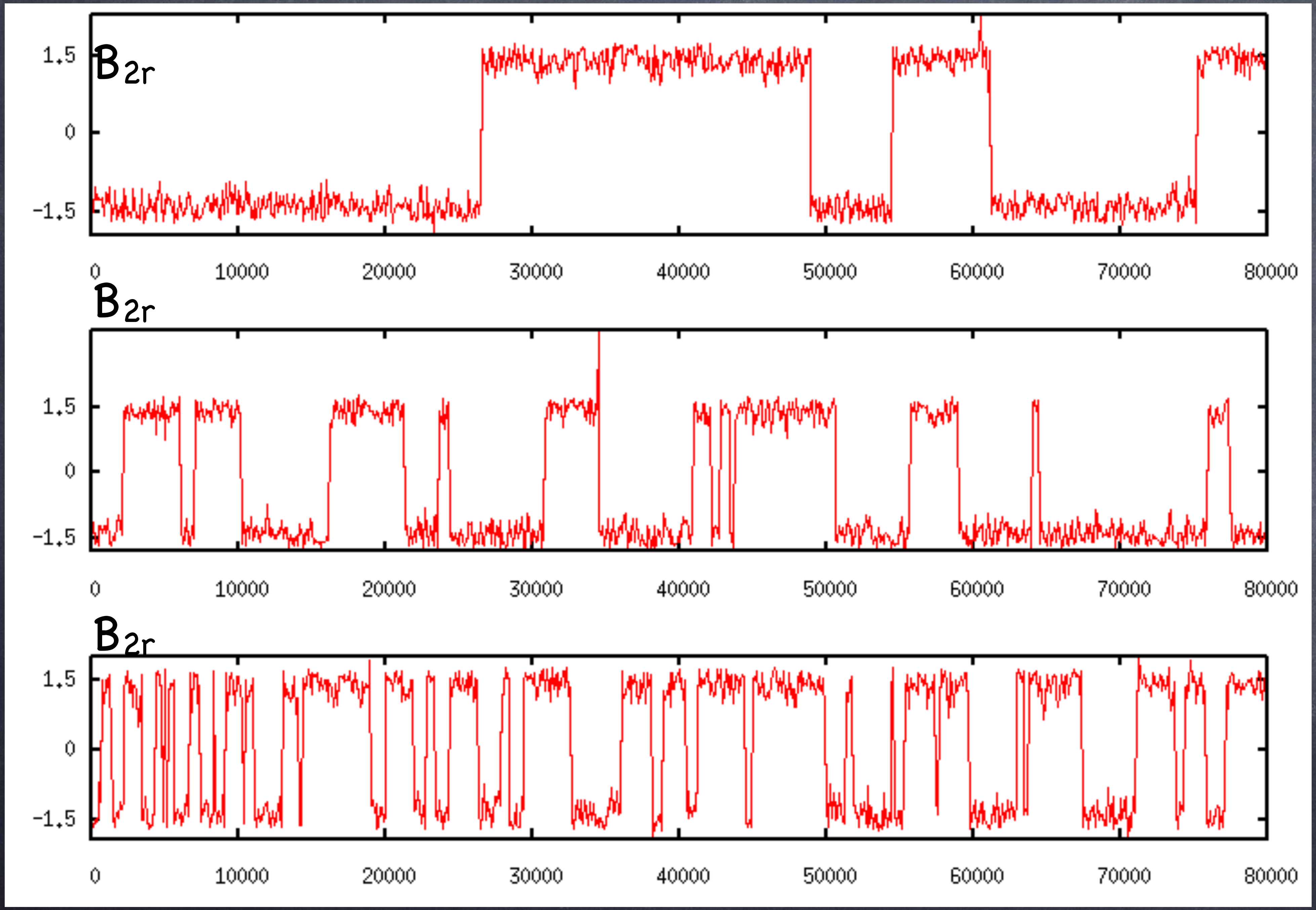
$$\beta - a_m(B_{2r}^2 - B_{2i}^2) - \nu_m k_2^2 = 0$$

This implies, for small ν_m, $\quad B_{2i} \sim 0 \quad B_{2r} \sim \pm \left[\frac{\beta}{a_m}\right]^{1/2}$

This is a "typical" argument obtained by using the theory of dynamical system and in particular bifurcation theory. It can be generalised and, in some cases, it can be proved to be rigorous.

Results from numerical simulations with $Re=10^7$ of the full non linear model

The equilibrium of B_{2r} depends on the turbulent energy transfer from the velocity to the magnetic field: STATISTICAL EQUILIBRIUM



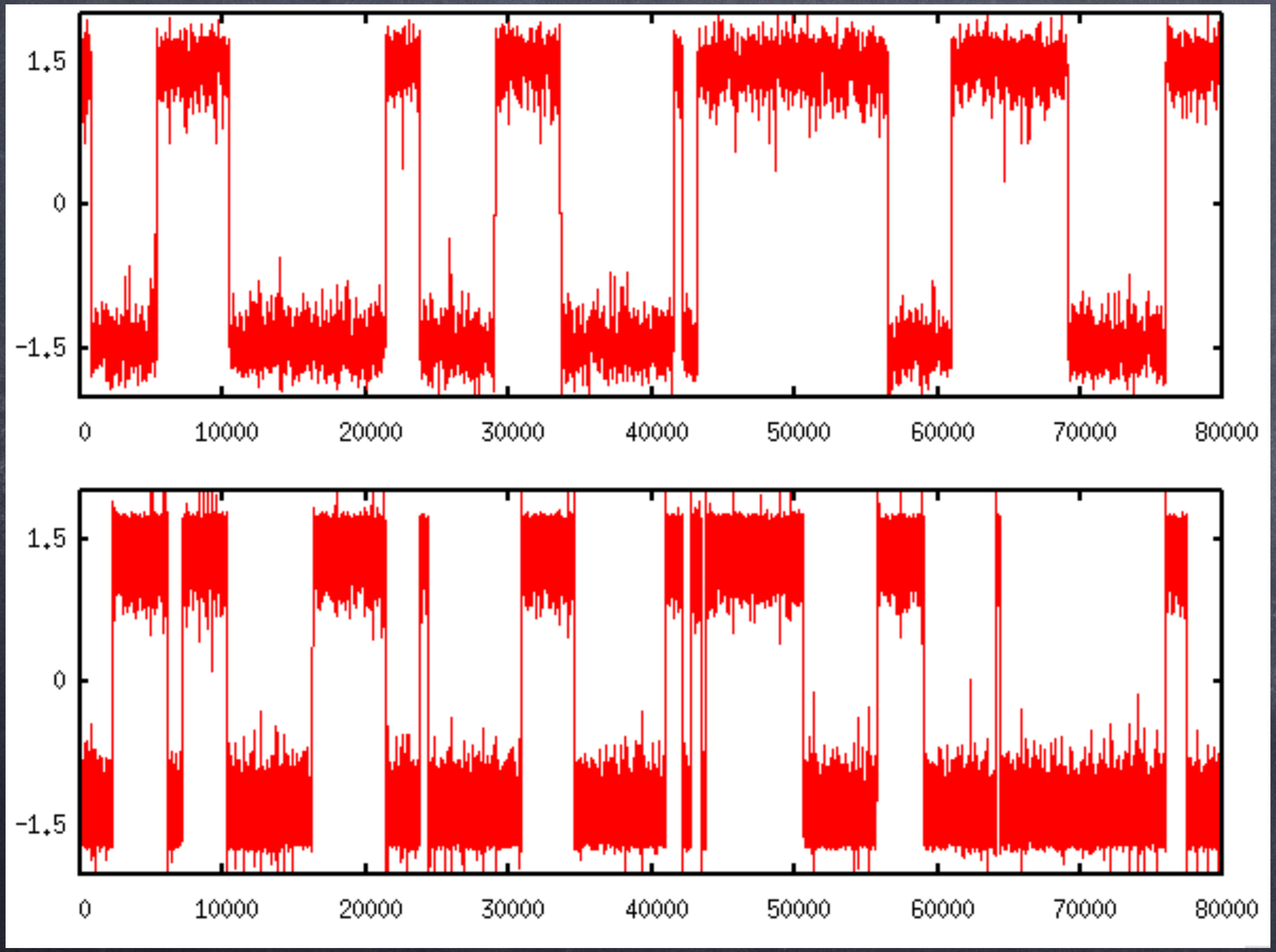
$\nu_m=0.00028$

$\nu_m=0.00026$

$\nu_m=0.00024$

We can approximate the dynamics of B_{2r} with the "standard" prototype model borrowed from dynamical system:

$$\frac{dB_2}{dt} = B_{2r}(B_0^2 - B_{2r}) + \sqrt{\varepsilon}dW$$



Double well potential

Nonlinear turbulent simulations

ν_m=0.00026

Since the equilibrium are "statistical equilibrium", their values change when an external forcing is applied, i.e. there is a non trivial response to external perturbations. This is not true for the simplified approach!

A simple example

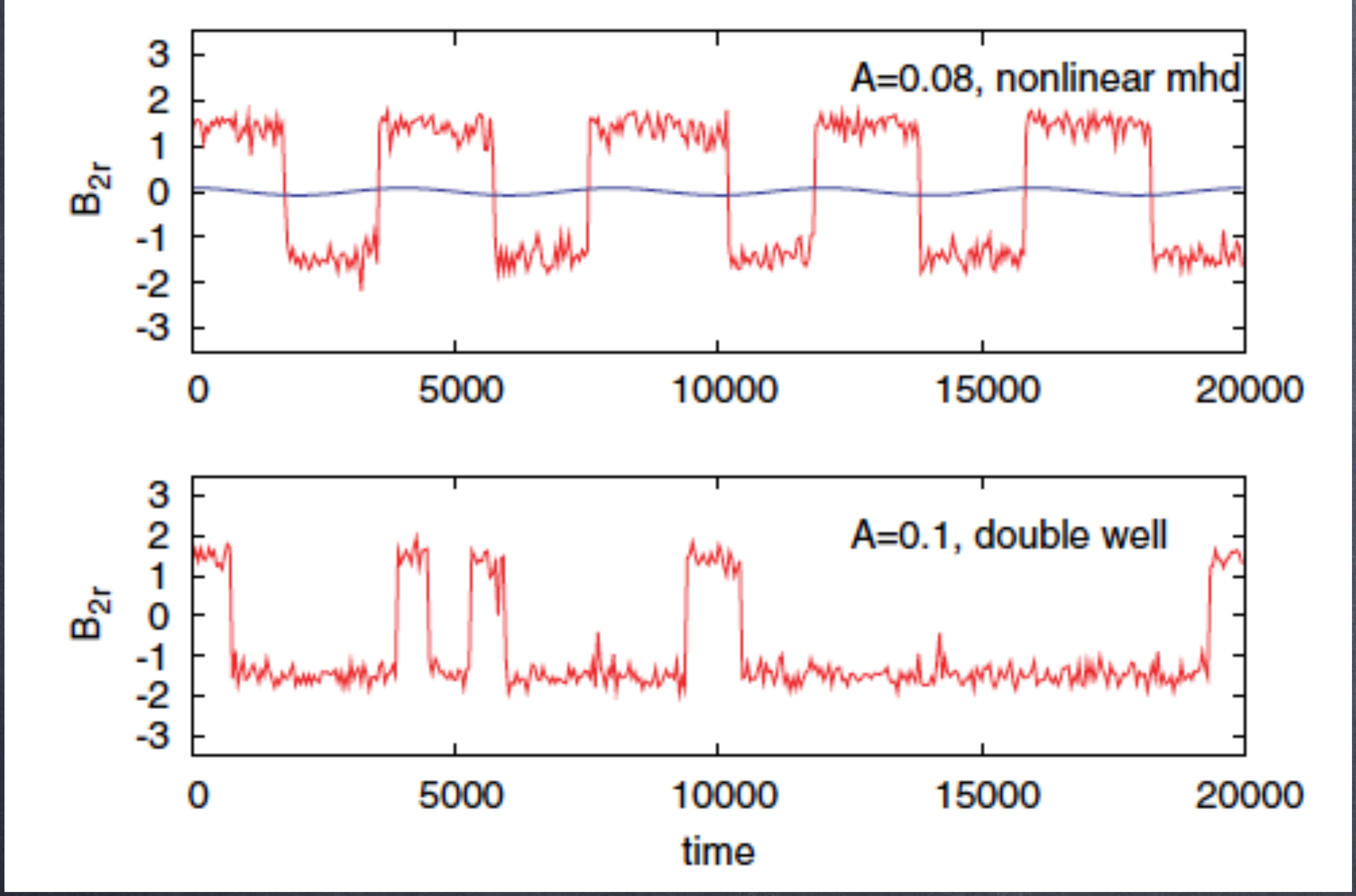
$$\frac{dB_{2r}}{dt} = B_{2r}(B_0^2 - B_{2r}^2) + Asin(\omega t) + \sqrt{\epsilon} dW(t)$$

$$\frac{dB_2}{dt} = \frac{i}{3}[\Phi_2(u,B) - \Phi_2(B,u)] - a_m B_2^3 - \nu_m k_2^2 B_2 + Asin(\omega t)$$

plus the full non linear model

$$\frac{du_n}{dt} = \frac{i}{3}(\Phi_n(u,u) - \Phi_n(B,B)) - \nu k_n^2 u_n + f_n \ ,$$
$$\frac{dB_n}{dt} = \frac{i}{3}(\Phi_n(u,B) - \Phi_n(B,u)) - \nu_m k_n^2 B_n \ ,$$

The equilibrium depends on the fluctuations around it (very complex situation) for the full non linear model. The external forcing changes the equilibrium and the fluctuations !! This is not true for our "simplified" model



Full non linear model

ν_m=0.00026

Simple dynamical model

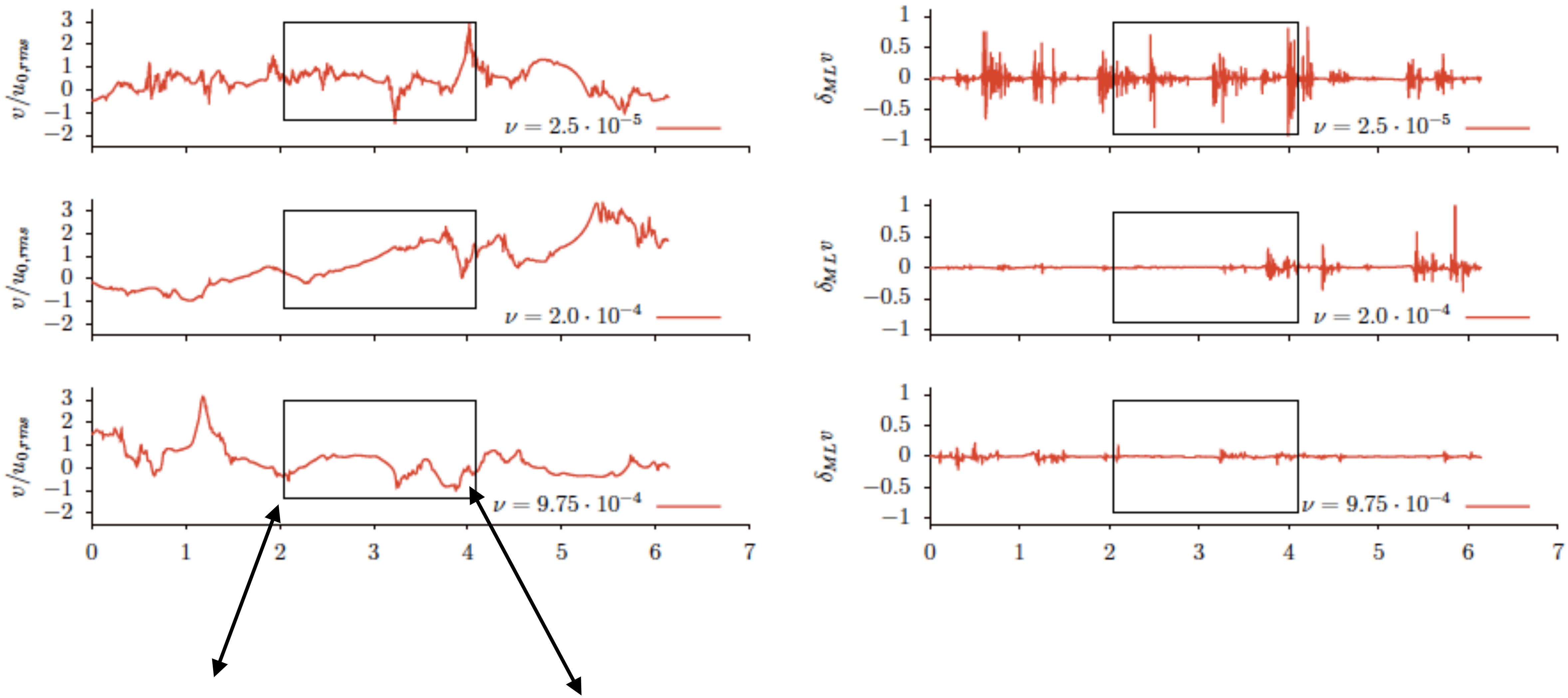
Different response functions for the same external forcing !

In this case, a dynamical system approach fails to reproduce the dynamics.
This is a "generic" features in many complex systems.

Machine learning and turbulence

A Corbetta[a], V Menkovski[b], RB[c] and F Toschi, Science 2021, in press

Velocity experienced by a lagrangian particle.



1 eddy turnover time

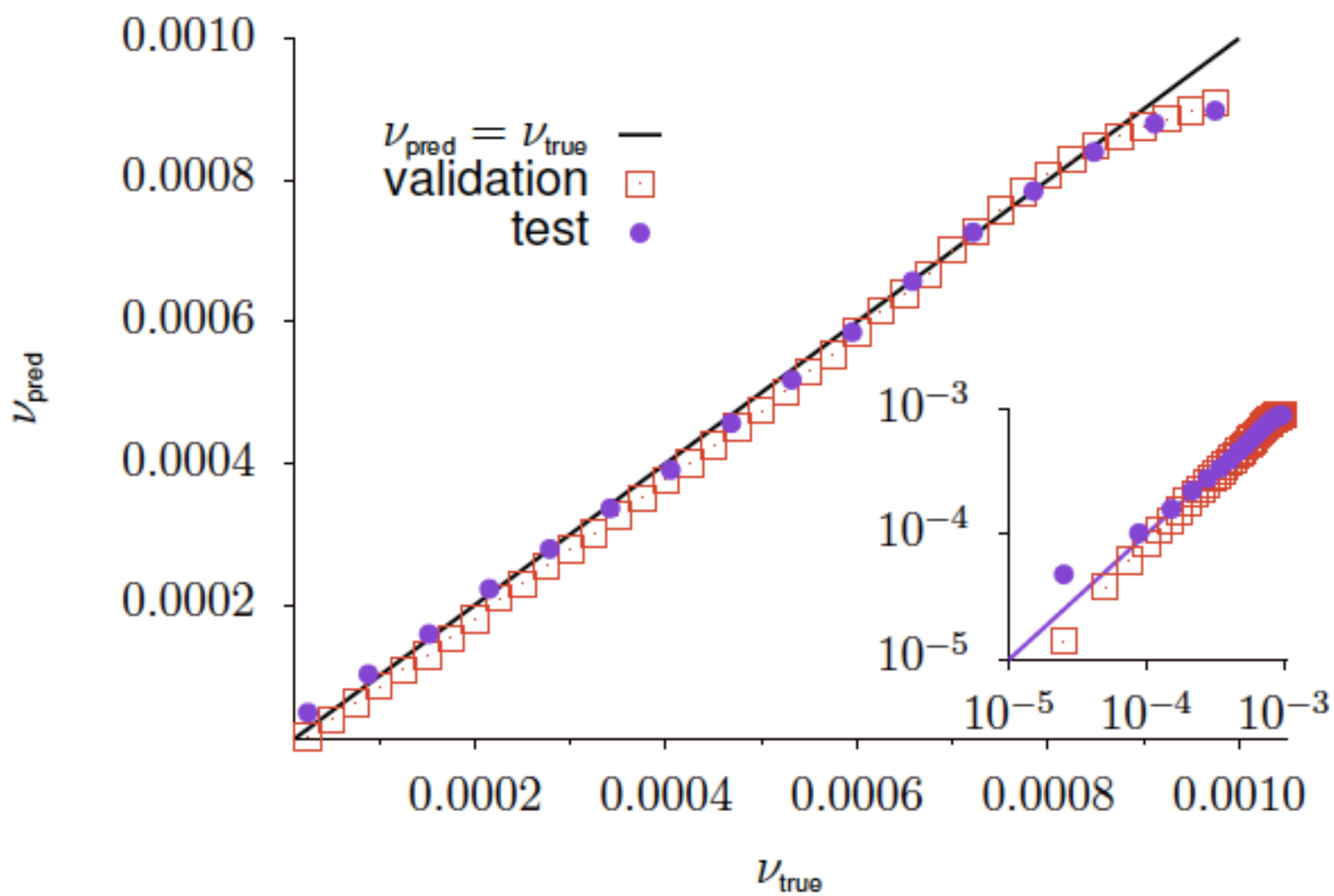
$$\langle \delta v(\tau)^2 \rangle = A\epsilon\tau \qquad \tau \gg \tau_\eta$$

$$\langle \delta v(\tau)^2 \rangle = B\nu^{-\alpha}\tau^2 \qquad \tau \le \tau_\eta$$

The constant A and B and the exponent α are known

Machine learning

Not surprising. Upon averaging over many validation test, we obtain the result are good !



The "standard approach".

Let Re the Reynolds number of our system. In order to compute Re given the velocity signal we can proceed as follows

1) compute

$$C = lim_{\tau\to 0}\frac{1}{v_{rms}^2}\left\langle\frac{\delta v(\tau)^2}{\tau^2}\right\rangle \qquad Re = \frac{v_{rms}L}{\nu}$$

We assume to know both v_{rms} and L. The knowledge of ν is equivalent to the knowledge of Re

The average <...> is done over several eddy turnover times:

$$\langle\ldots\rangle = \frac{1}{N}\Sigma_i \qquad C = \langle C_i\rangle = \frac{1}{N}\Sigma_i C_i$$

Where C_i is the value of "C" averaged over 1 eddy turnover time.

2) From the turbulence theory we know $C = DRe^{\alpha}$

where D depends on the geometry (it is supposed to be known) and α can be computed using the multifractal theory. The error of Re depends on the error on C and it small for N large enough.

For a flow at Re=10^5 and eddy turnover time 1 sec, after 5 minutes we reach an accuracy of 30%.

Suppose that we use the previous procedure with only 1 eddy turnover time, N=1

$$\frac{C_i}{C} = \frac{Re_p^{\alpha}}{Re^{\alpha}}$$

where **Re_p** is the "**predicted value**" of Re with just **1 eddy turnover time.**

From each eddy turnover time we obtain different values of **Re_p** because of fluctuations

$$Re_p^{\alpha} = Re^{\alpha_i}$$

multifractal fluctuations

$$\frac{Re_p^{\alpha}}{Re^{\alpha}} = Re^{\alpha_i - \alpha}$$

α_i are fluctuations around α due to the **fluctuations of the dissipation scale**

$$P[\alpha_i] = Re^{-b(\alpha_i - \alpha)^2} = exp[-b(\alpha_i - \alpha)^2 log(Re)]$$

Simplified multifractal (lognormal) pdf for α_i $\quad b \leq 1$

We can now compute our error over 1 eddy turnover time

$$P[\alpha_i] = Re^{-b(\alpha_i - \alpha)^2} = exp[-b(\alpha_i - \alpha)^2 log(Re)]$$

$$\sqrt{\langle(\alpha_i - \alpha)^2\rangle} \equiv \sigma = \sqrt{\frac{1}{2b log(Re)}}$$

$$\frac{Re_p^\alpha}{Re^\alpha} = Re^{\alpha_i - \alpha} \quad \Rightarrow \quad \alpha \;\; log\left(\frac{Re_p}{Re}\right) = (\alpha_i - \alpha) log(Re)$$

$$\xi = log\left(\frac{Re_p}{Re}\right) = \frac{\alpha_i - \alpha}{\alpha} log(Re)$$

$$\delta\xi \sim \frac{\sigma}{\alpha} log(Re) \sim \frac{1}{\alpha}\sqrt{\frac{log(Re)}{2b}}$$

error over 1 eddy turnover times

We can now compute our error over 1 eddy turnover time

$$\delta\xi \sim \frac{\sigma}{\alpha} log(Re) \sim \frac{1}{\alpha}\sqrt{\frac{log(Re)}{2b}}$$

error over 1 eddy turnover times

$$\delta\xi_N \sim \frac{1}{\alpha}\left[\frac{log(Re)}{2bN}\right]^{1/2}$$

error over N eddy turnover times

An example. For Re = 10^5 $\delta\xi \sim 5$

For a flow with eddy turnover time 1sec, after 5 minutes we have $\delta\xi_N \sim 0.28$

$$\delta\left(\frac{Re_p}{Re}\right)_N \sim e^{\delta\xi_N} \sim 1.32$$

(30% accuracy)

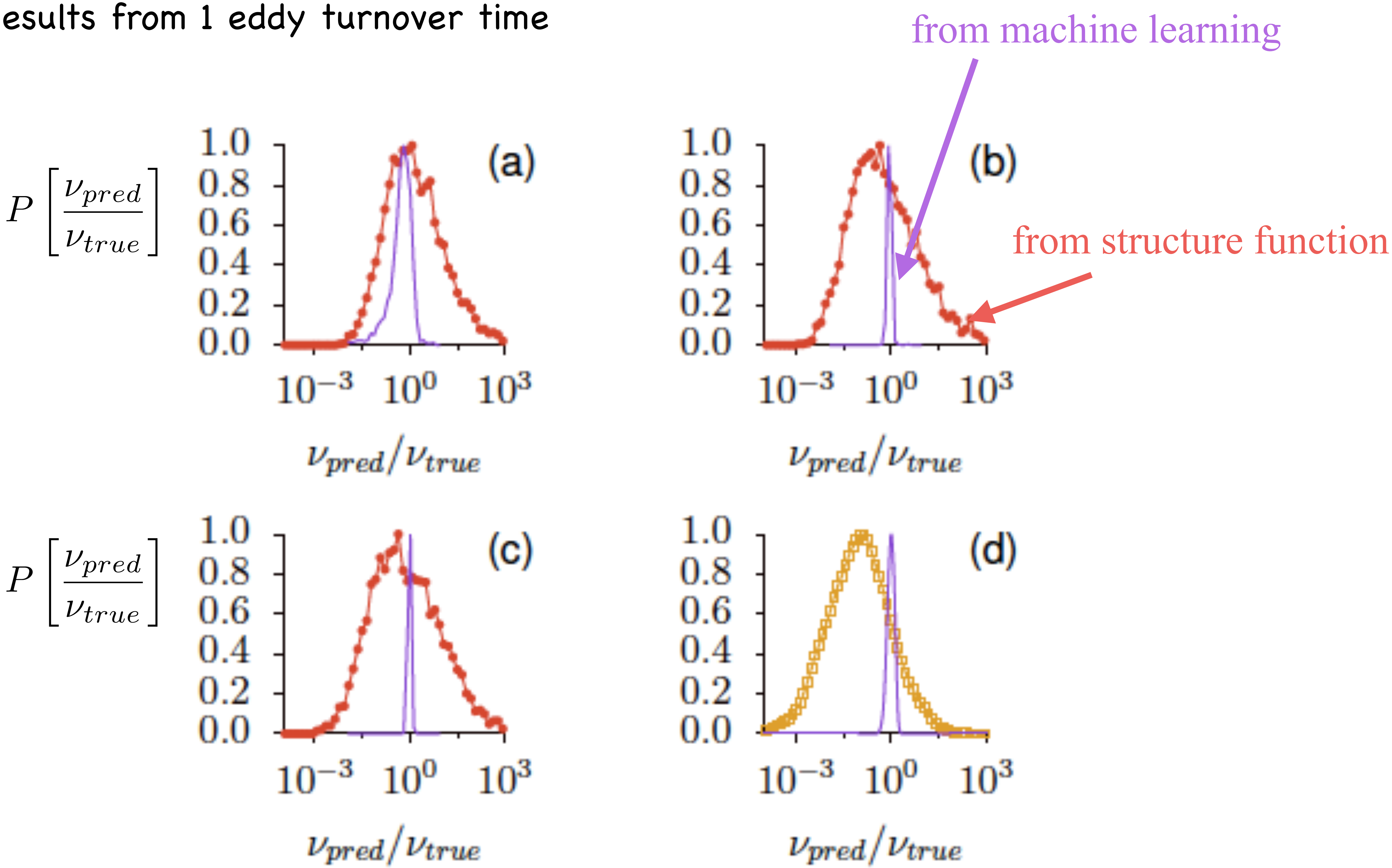
Using 1 eddy turnover time we have

$$\delta\left(\frac{Re_p}{Re}\right) \sim e^{\delta\xi}$$

$$\boxed{\delta\left(\frac{Re_p}{Re}\right) \sim 100}$$

How does the machine learning perform?

Results from 1 eddy turnover time



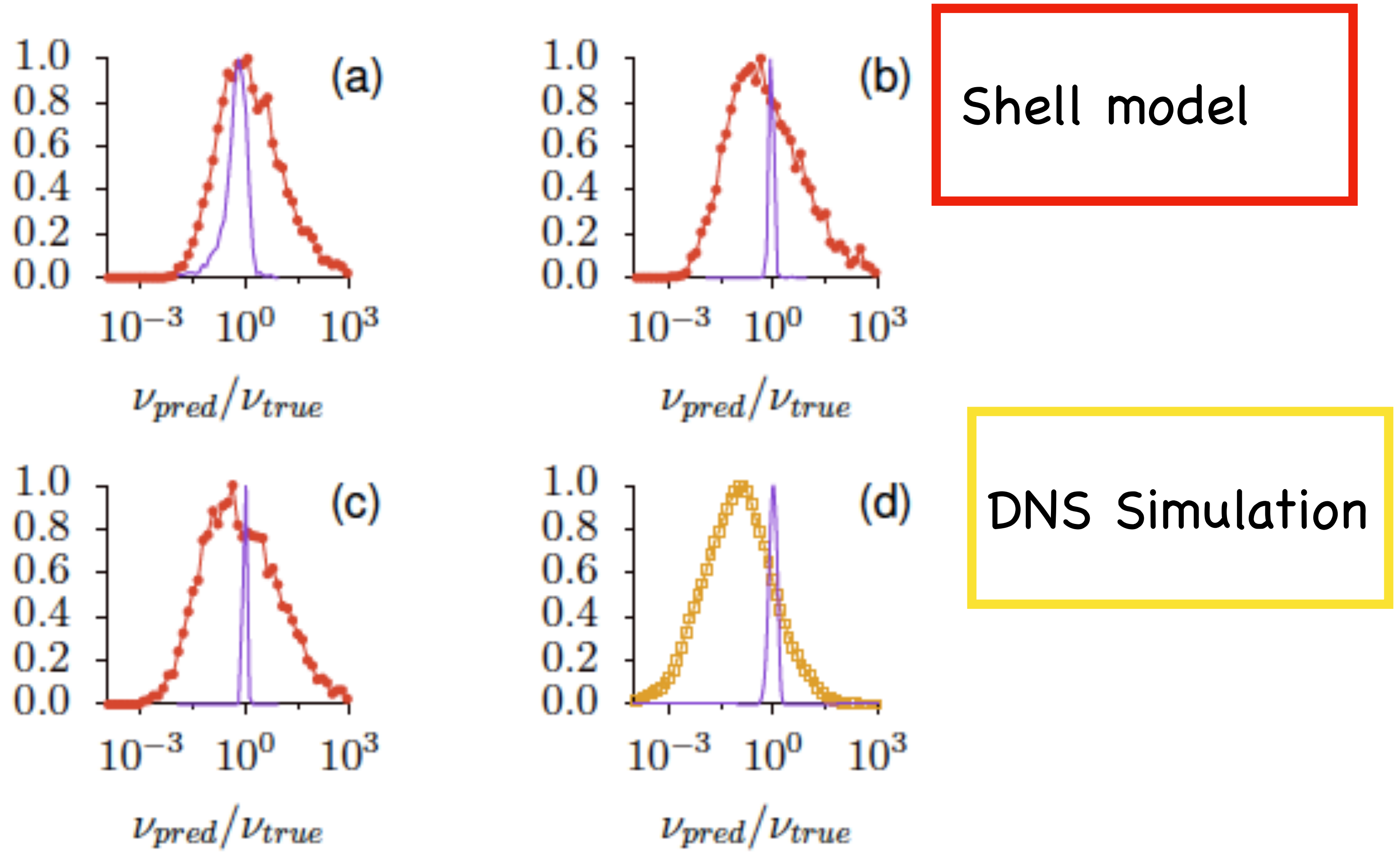
Using machine learning (pattern recognition) we obtain error of 10% !!

NOTE: machine learning was trained by using signal for different viscosity using a shell model and the velocity signal

$$U(t) = \Sigma_n Real(u_n(t))$$

which is known to behave as a lagrangian particle.

After training, machine learning was applied to data from DNS simulations.



Why machine learning is performing so well?

A tentative explanation.

Machine learning uses pattern recognition algorithm able to "learn" small fluctuations at different time scales and their correlations within the signal.

This is equivalent to "learn" the multifractal fluctuations in the inertial range and to learn how they correlates at different scales.

Machine learning, in this particular case, is able to sample the multifractal "spectrum" better then any naive statistical approach based on "standard tools".

More work on this interpretation is needed.

Avalanches in turbulence

I. Castaldi, RB, F. Toschi and J. Trumpert, in preparation.

Let us consider a shell model of turbulence.

$k_n = 2^n k_0 \rightarrow u_n$ complex variable

non linear interactions with required invariants and scale symmetry

$$\frac{du_n}{dt} = i\left(ak_{n+1}u_{n+2}u^*_{n+1} + bk_n u_{n+1}u^*_{n-1} - ck_{n-1}u_{n-1}u_{n-2}\right) - \nu k_n^2 u_n + f_n\,,$$

a=1, b= -0.4, c= -0.6

Let chose a particular form of f_n

$$f_1 = \frac{(1+i)F}{u_1^*} \qquad f_2 = \frac{(1+i)F}{u_2^*}$$

$E \equiv \Sigma_n |u_n|^2$ energy

$\epsilon = \nu \Sigma_n k_n^2 |u_n|^2$ energy dissipation

$$P \equiv \Sigma_n |f_n u_n^*| = 4F$$

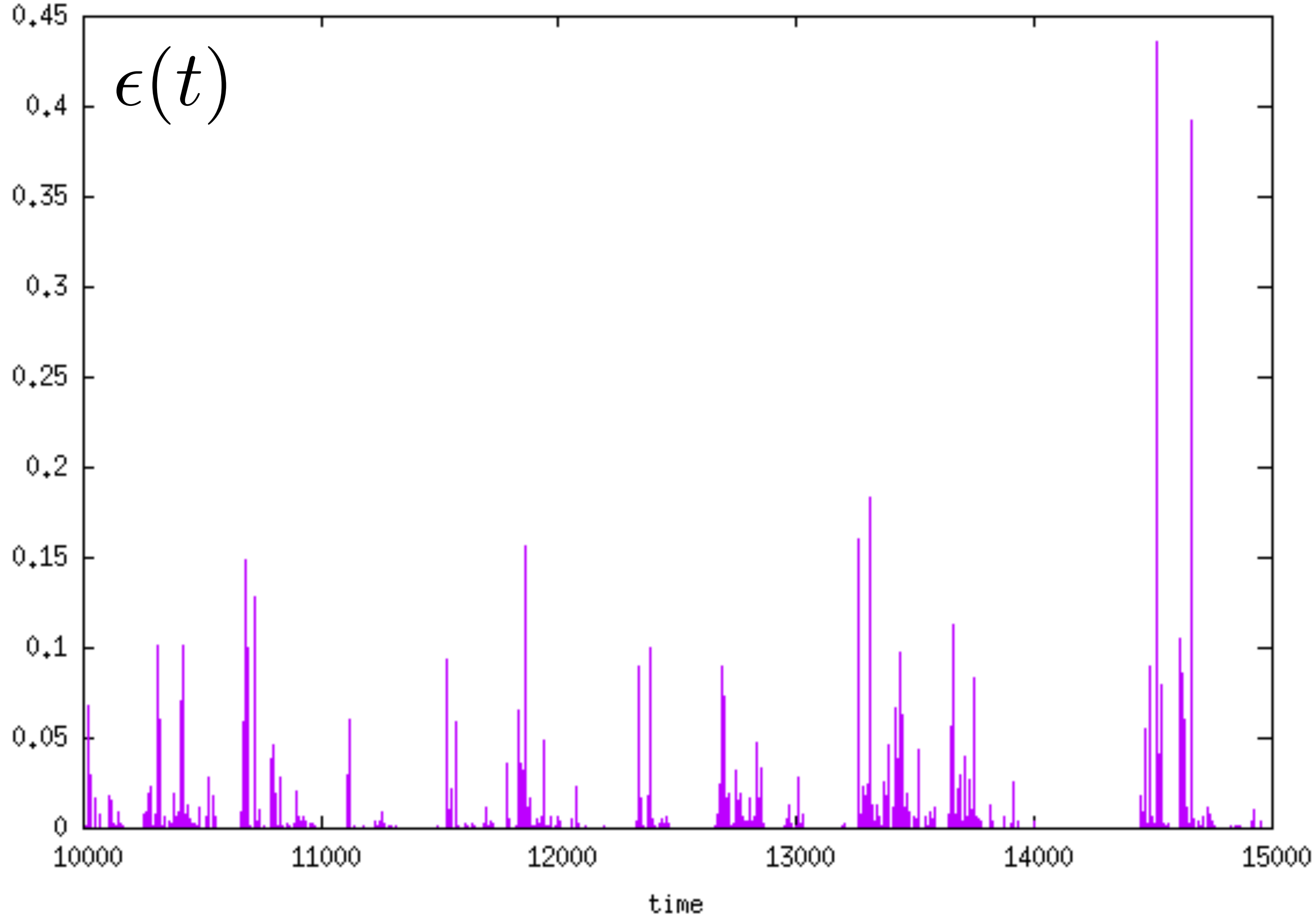
Rate of energy input

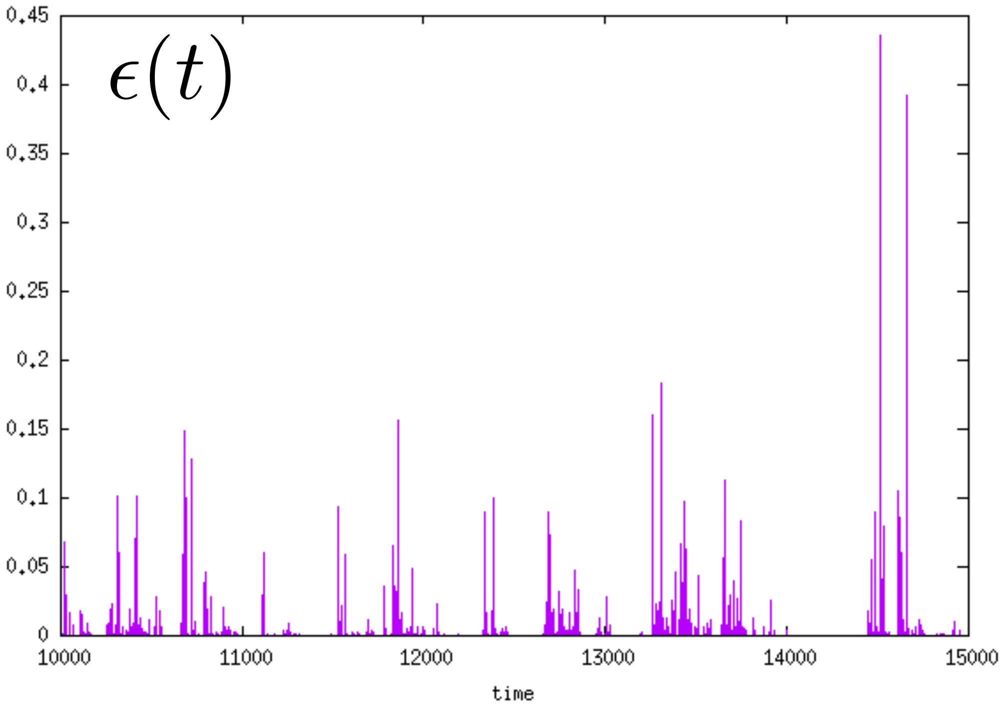
$$\frac{dE}{dt} = P - \epsilon(t)$$ rate of energy dissipation is time dependent

$$if \;\; \epsilon(t) \leq P \rightarrow \frac{dE}{dt} > 0$$ The energy grows in time

$$if \;\; \epsilon(t) > P \rightarrow \frac{dE}{dt} < 0$$ The energy is dissipated

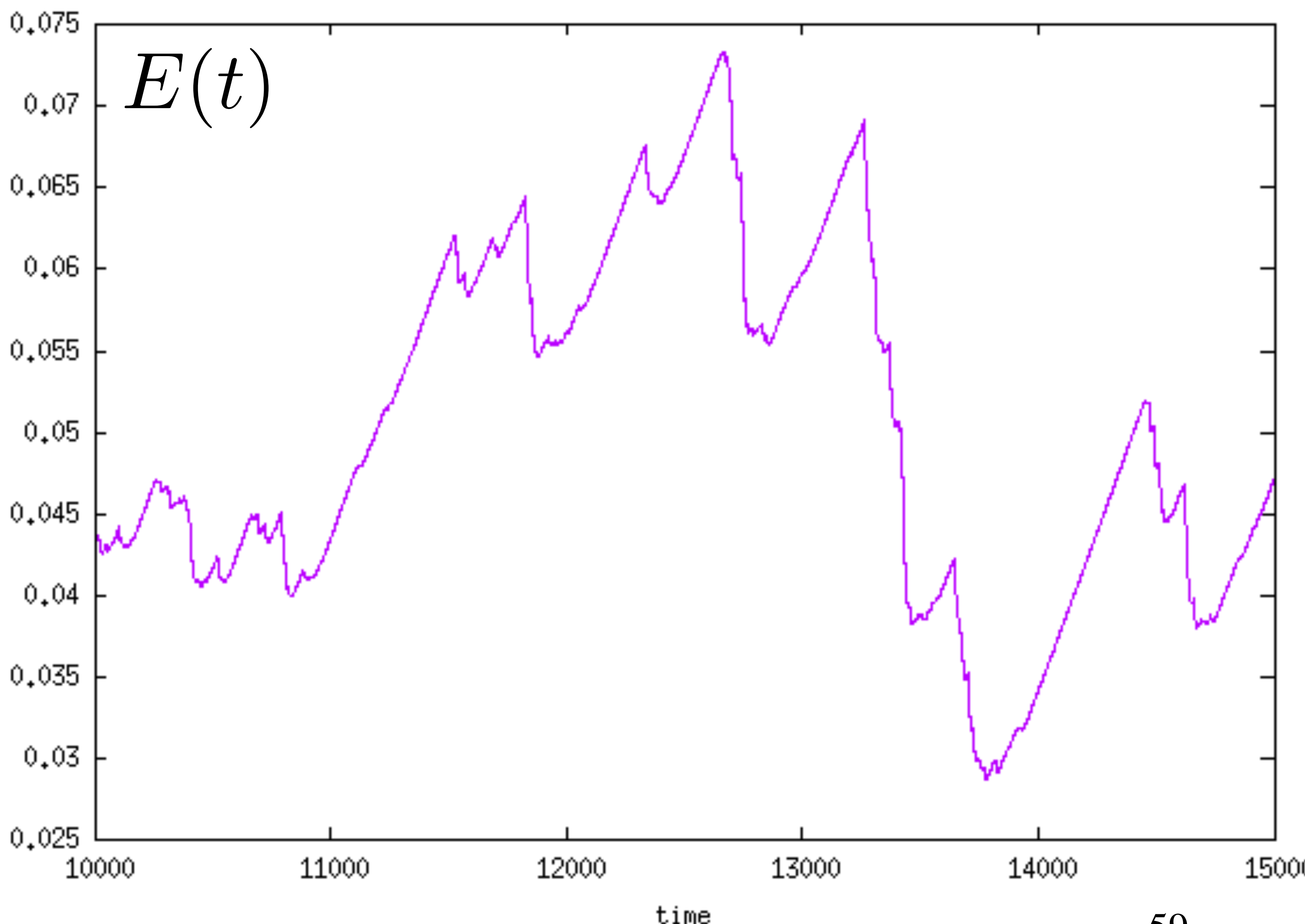
energy dissipation





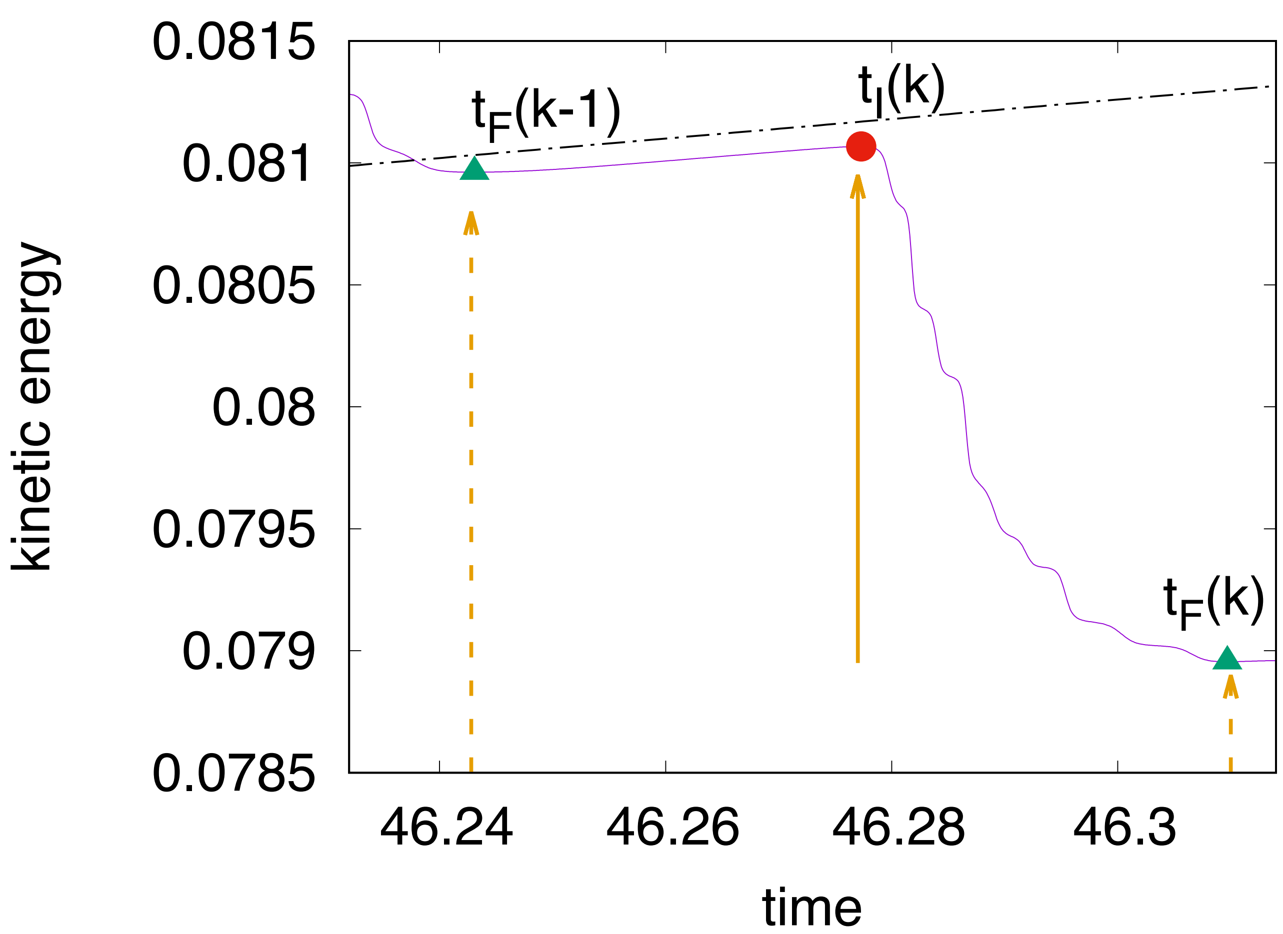
avalanche like events

Time scale of dissipation much shorter than the time for energy to grow.



Questions to be investigated: avalanche size, time between avalanches

These are general questions which are considered to be crucial in a number of physical problems (not only turbulence!).



t_i(k) = time at which dE/dt change sign from positive to negative

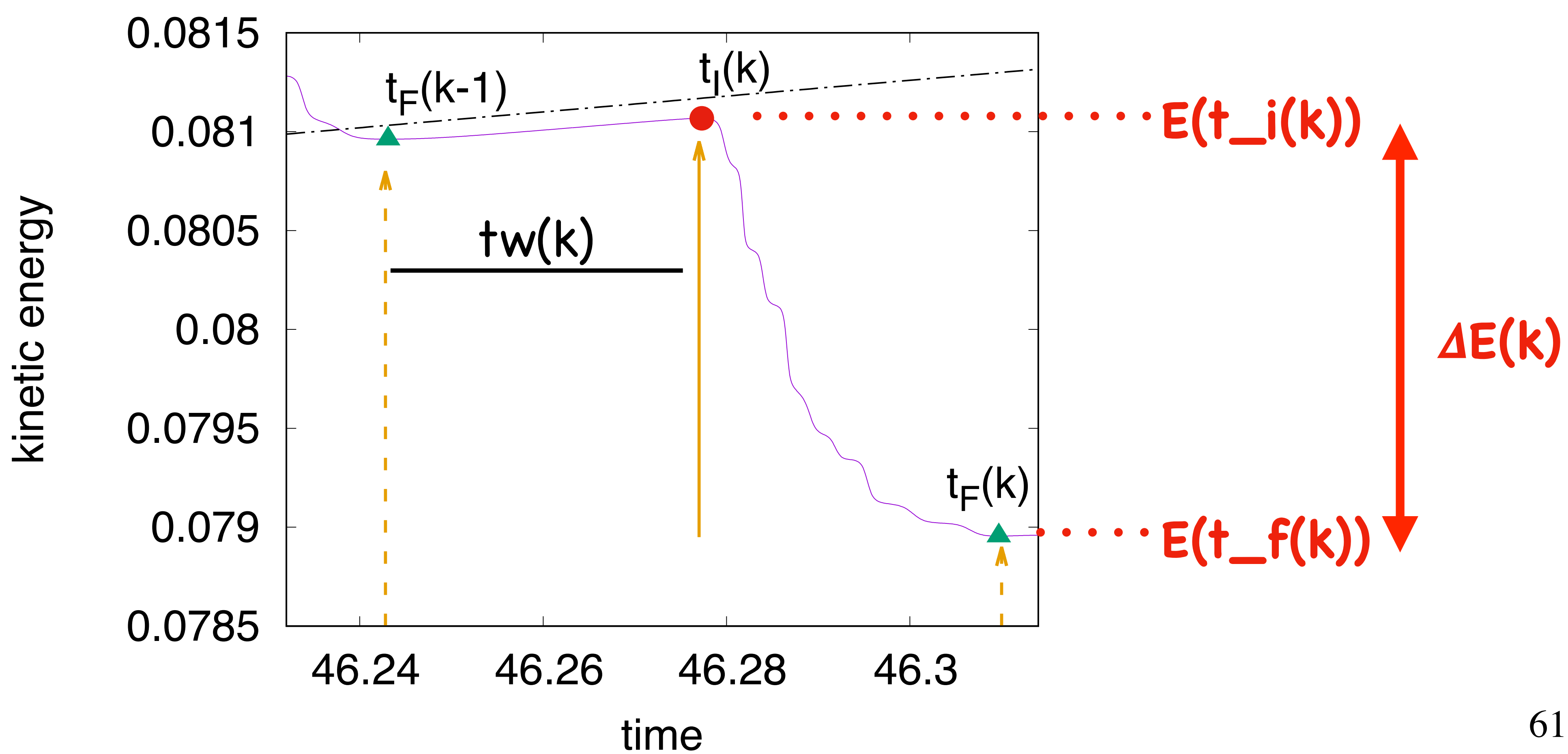
t_f(k) = time at which dE/dt change sign from negative to positive

To analyse the "avalanche" dynamics we must first define what we mean by avalanche and which may be the relevant informations to be studied.

At this stage, our previous knowledge of turbulence is not relevant.

1) Size of the event **ΔE(k) = E[t_i(k)]-E[t_f(k)]**

2) Inter event time or waiting time between events **tw(k)= t_i(k)-t_f(k-1)**



1) Size of the event **ΔE(k) = E[t_f(k)]-E[t_i(k)]**

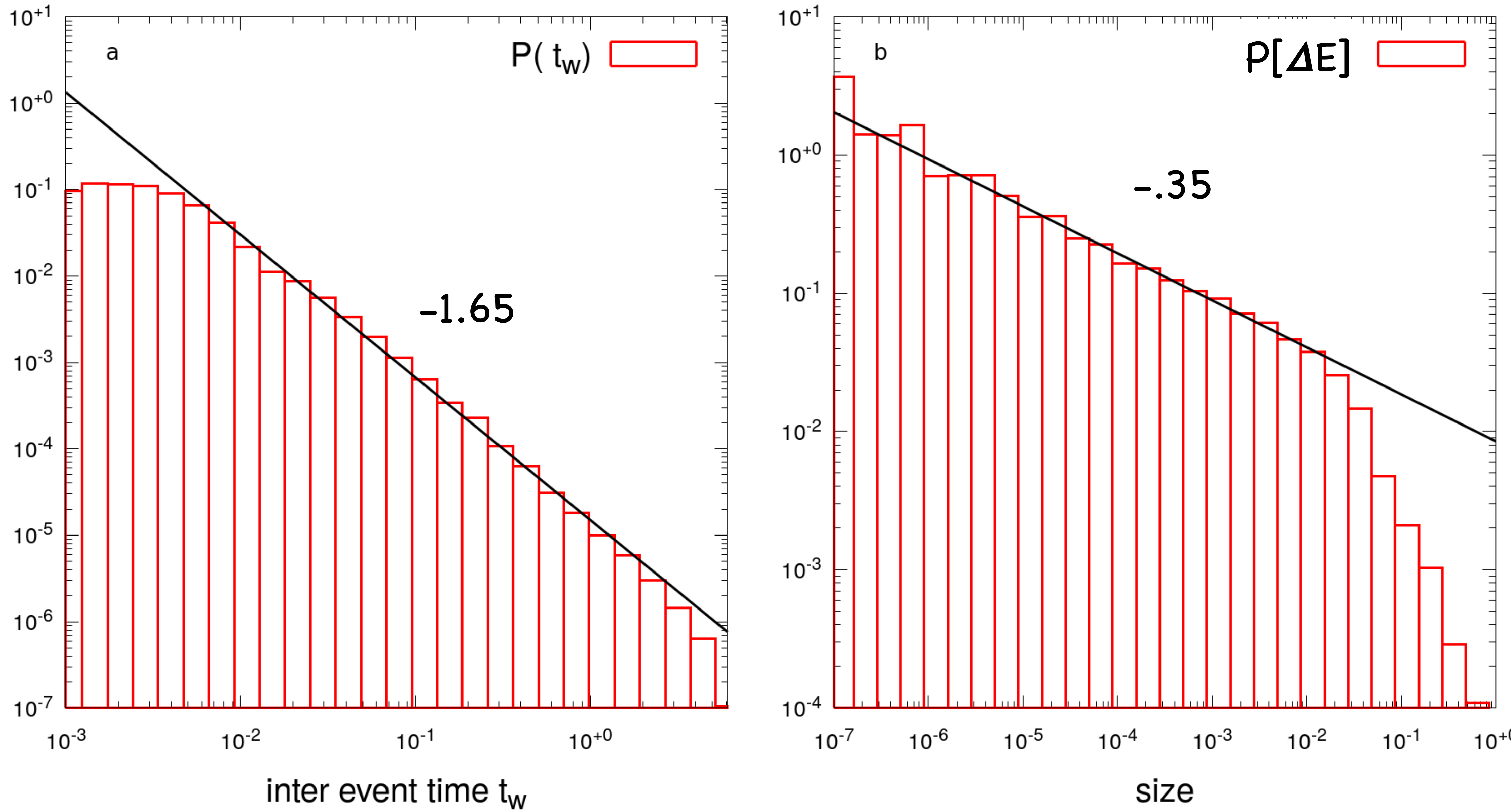
2) Inter event time or waiting time between events **tw(k)= t_i(k)-t_f(k-1)**

Then the basic question concerns the probability distribution of P[**ΔE(k)**] and P[**tw(k)**]

In seismology P[**ΔE(k)**] show a scaling behaviour known as the Guttenberg-Richter law.

For random uncorrelated processes P[**tw(k)**] is a Poisson distribution.

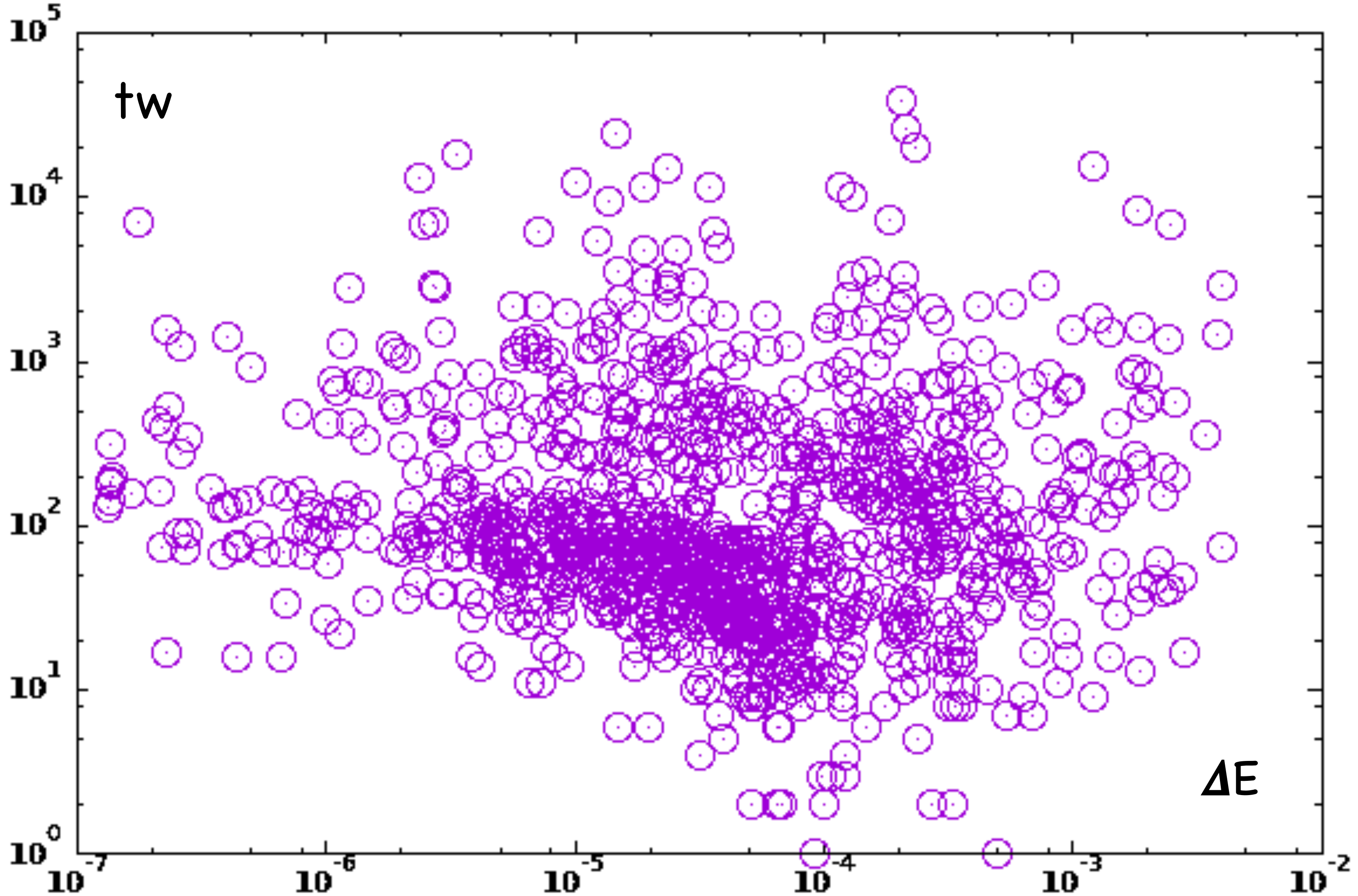
Both probability distributions of size and inter event time are scaling functions of their argument !



Scaling may not be so surprising !! After all we are looking at a turbulent flows where energy dissipation shows non trivial scaling properties in space (i,e, k_n) and time.

A non trivial observation

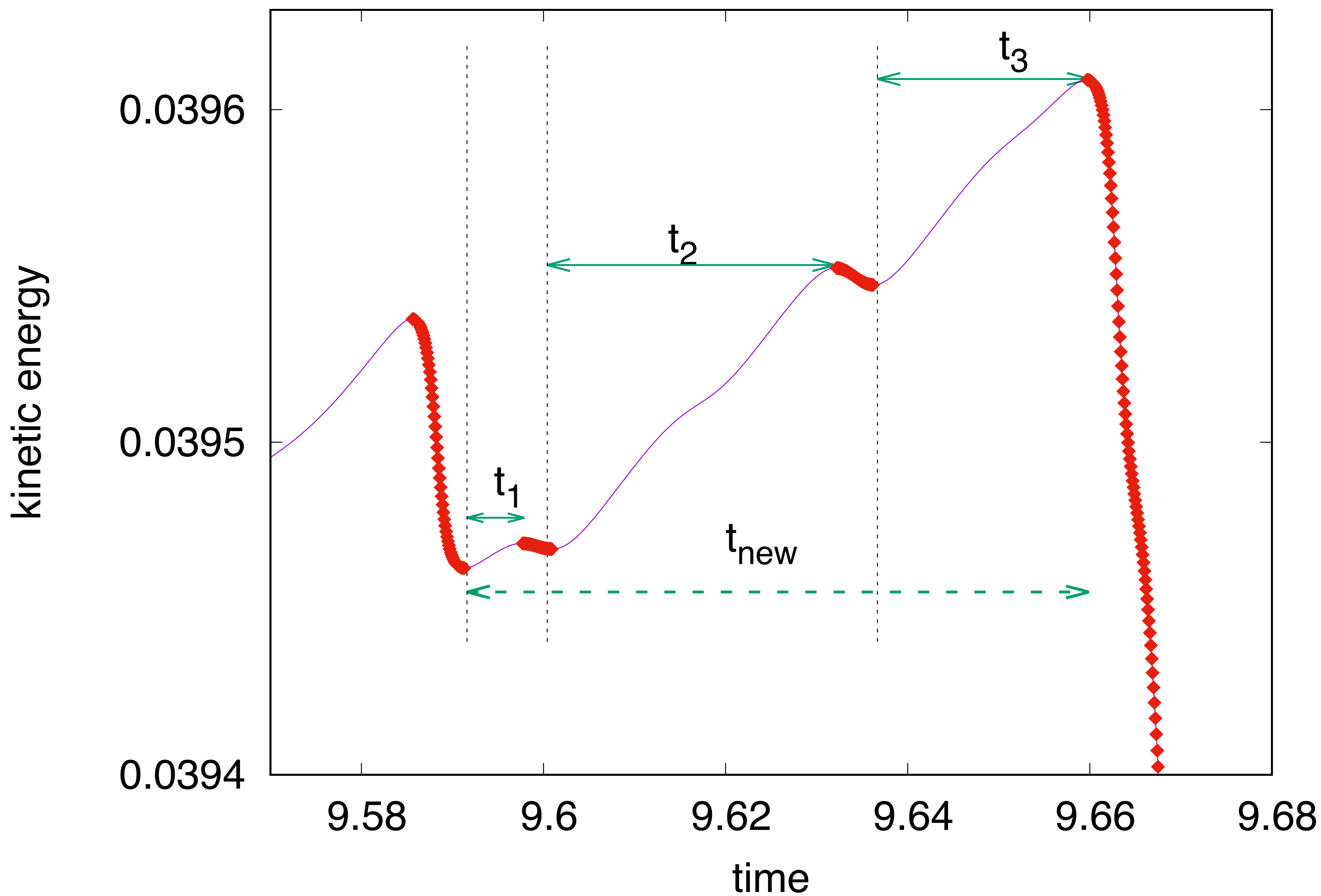
There are no correlations between size and inter event time



A closer look to P[tw].

ΔE is defined between some minimum value ΔE_0 and some maximum value ΔE_M

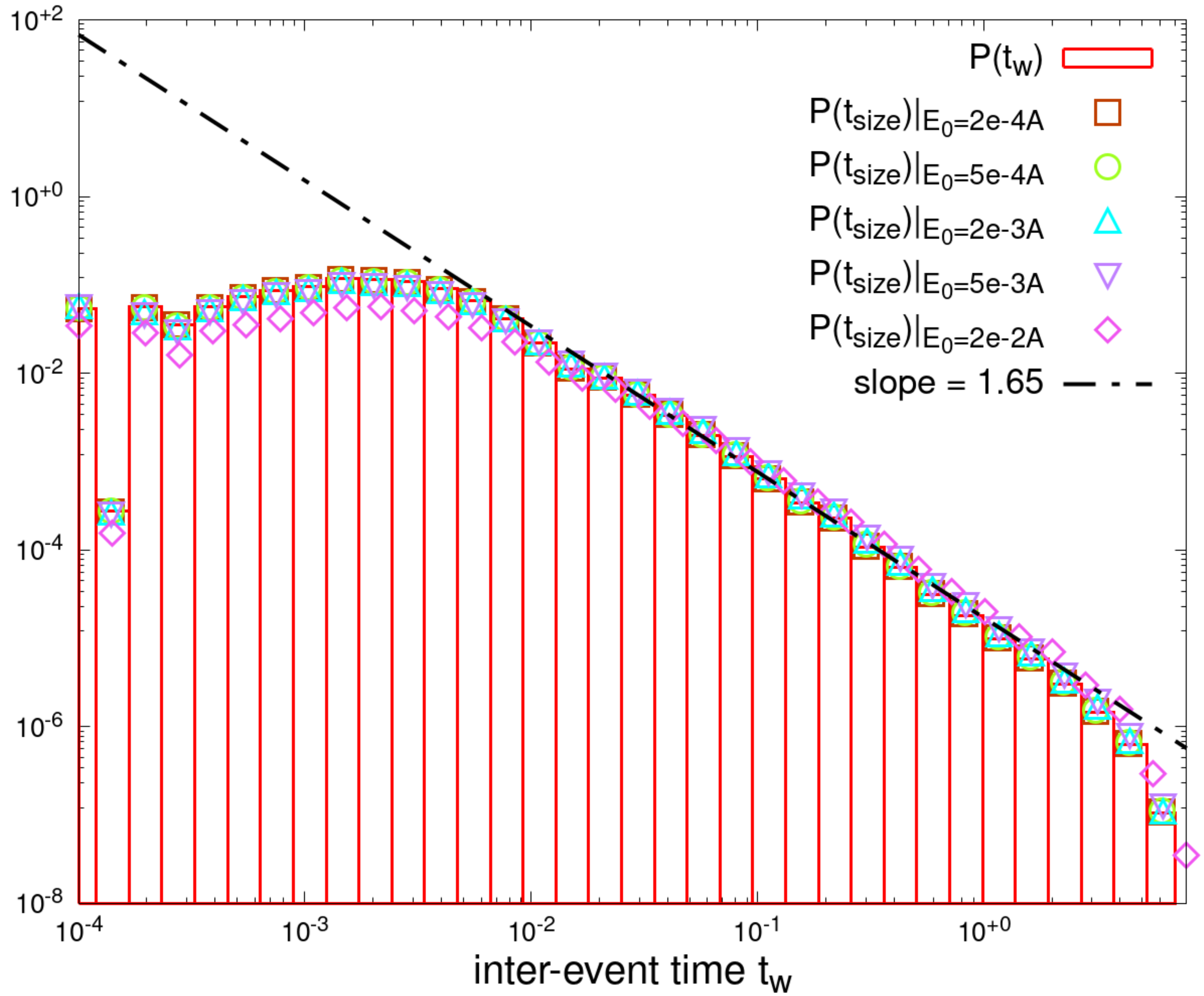
Let us now look at the inter event time between events whose size ΔE is larger than some threshold ΔE_t



The we can look at the probability distribution of tw in the "coarse grained" signal:

P[tw|ΔE>ΔE_t]

The non trivial results is that P[tw|ΔE>ΔE_t] does not change its functional form!



Probability theory tells us that there exists non trivial correlation among tw

ΔE_t is changing by 2 order of magnitudes.

A similar effect is also observed in earthquake dynamics (Corral 2004).

New and non trivial physics to learn?

Thanks for your attention