Recurrence quantification analysis approaches in complex systems dynamics

Reik V. Donner

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Dynamical systems

Dynamical systems can be classified according to different dimensions:

- Time-discrete vs. time-continuous
- Deterministic vs. stochastic (different mathematical frameworks, addressed by different model types and analysis techniques)
  - How to deal with deterministic dynamics corrupted by noise? Filter noise (averaging, time scale separation/decomposition,…)? Separate or combined model/analysis?
- Linear vs. nonlinear (functional type of dependencies among different variables or observations of the same variable at different points in time)
- For deterministic systems: conservative vs. dissipative (in a statistical mechanics sense: how does the state space volume covered by the system change with time)
- Also for deterministic systems: regular (fixed point/periodic/quasiperiodic vs. deterministic-chaotic)
  - Conservative (Hamiltonian) dynamics: mixed state space with coexisting domains
  - Dissipative systems: attractors and repellors with their respective (time-forward or time-backward) basins of attraction, including possible multistability

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Nonlinear time series analysis

Overview (no exhaustive classification, there exist more methods!)

Duality between two approaches based on dynamical systems theory and information theory/statistical mechanics

Single time series analysis – for diagnosing/classifying the dynamics of a macroscopic observable

- State-space based methods (Correlation dimension, recurrence plots)
- Information-theoretic methods (Entropies, complexity measures)

Bi- and multivariate analysis – for diagnosing/classifying/quantifying dependencies among variables

- State-space based methods (Single- and multi-channel singular spectrum analysis, Cross-/joint recurrence plots)
- Information-theoretic methods (Mutual information/redundancies, transfer entropy)

Ultimate goals of both aspects:

- Selection of a proper model class/empirical model development
- Predicting the dynamical behavior

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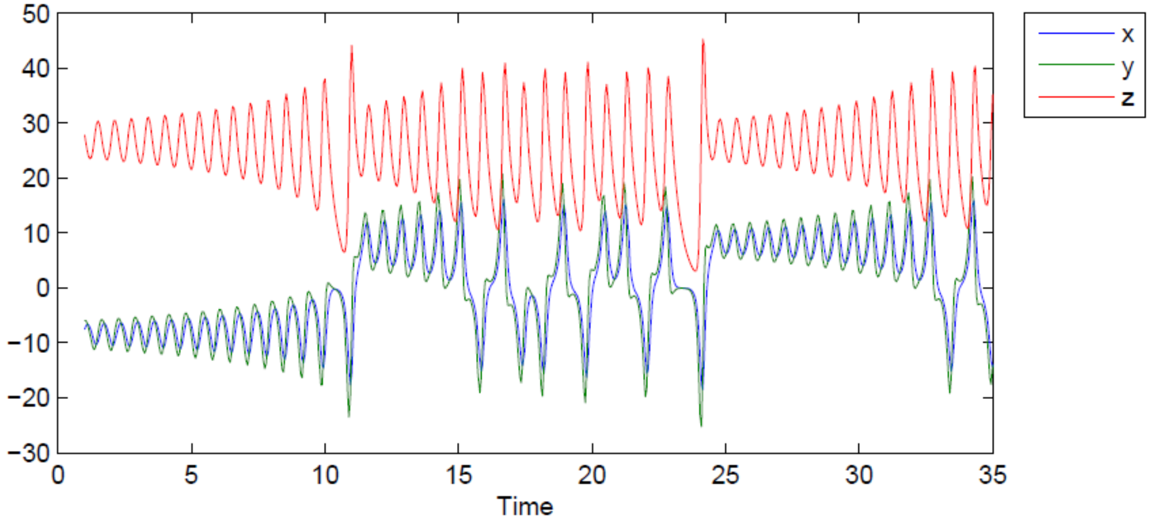
Phase/state space concept

Dynamical systems typically have more than one relevant state variable

$\Rightarrow$ *n*-dimensional state space spanned by these variables



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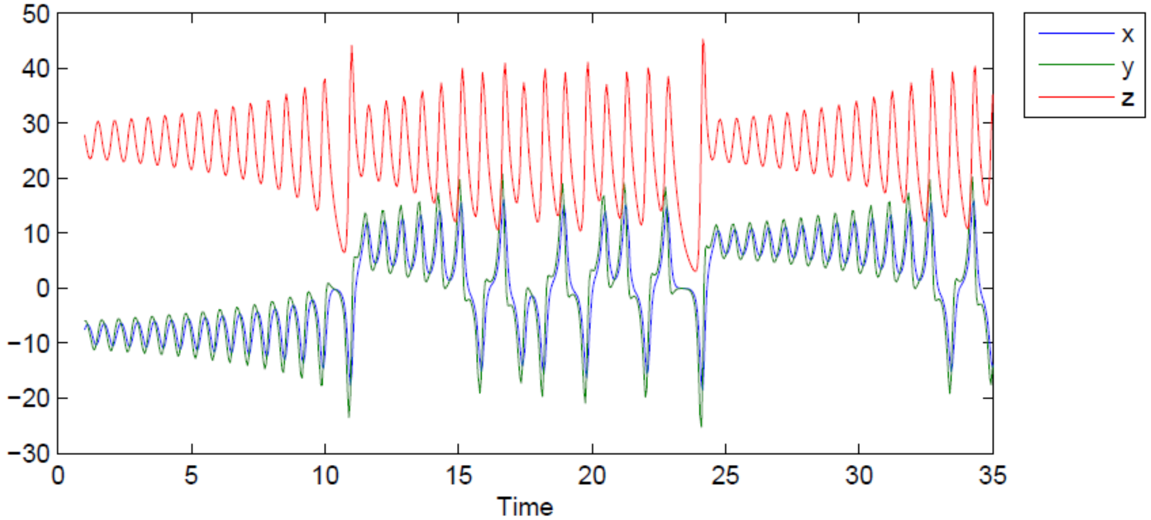
Time-delay embedding

Dynamical systems typically have more than one relevant state variable

But: one can often only observe/measure some component!



$\Rightarrow$ How to estimate the true (topological) dimensionality of the system?

Time-delay embedding

Takens embedding theorem: Chaotic attractors can be reconstructed from a single component (a time series) using proper embeddings

Common approach: time-delay embedding

$$\boldsymbol{x}(t) = (x(t), x(t-\tau), \ldots, x(t-(m-1)\tau)$$

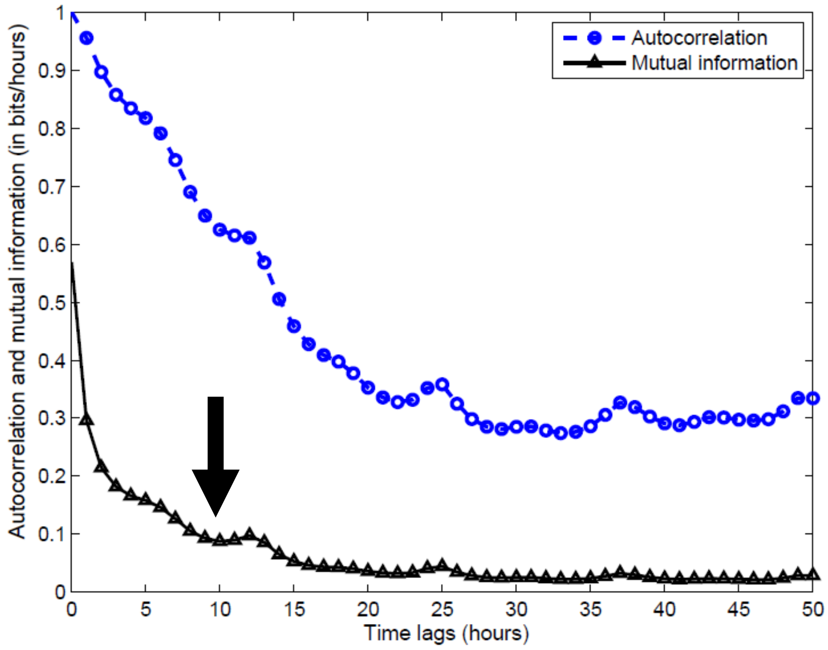
Embedding delay τ: identify independent components in time

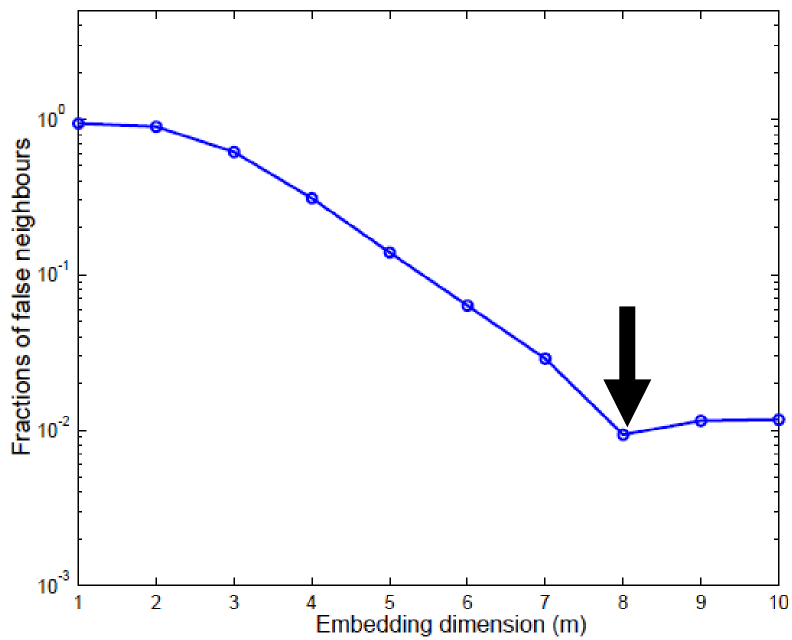
- Linearly independent: first root of auto-covariance function (difficult especially for non-stationary and long-range dependent data); alternatively: de-correlation time
- Statistically least dependent: first minimum of mutual information function

Embedding dimension m: false nearest-neighbor method

Time-delay embedding

Example: storm surge data *(Siek et al., Nonlin. Proc. Geophys., 2010)*





Time-delay embedding

Theoretical considerations: for addressing all relevant "directions" of the dynamical system under study, an embedding dimension of $m=2D+1$ or higher theoretically guarantees complete unfolding of the trajectory (with D being the topological dimension of the system)

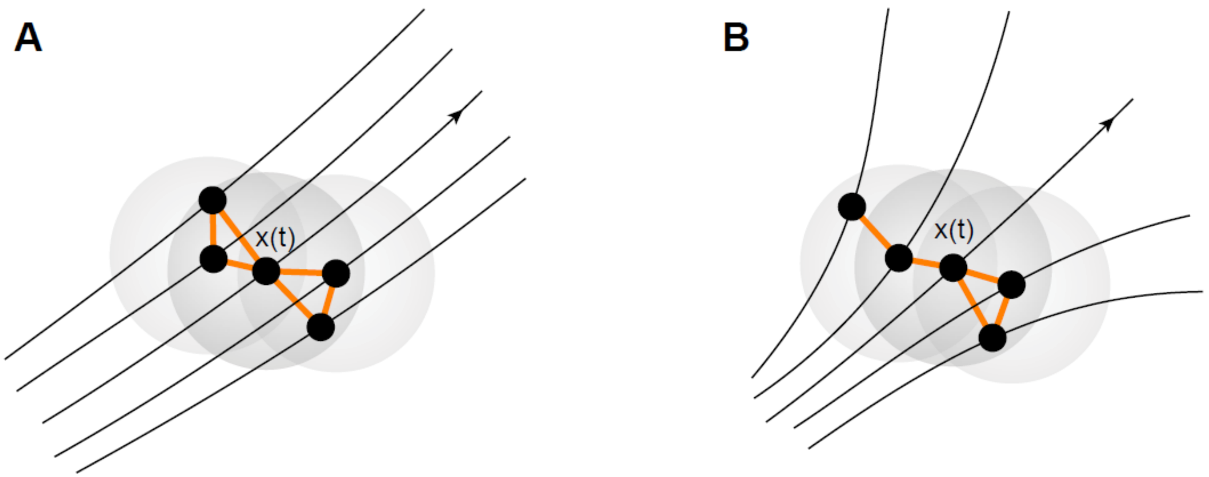
Practically, $m=D$ is usually sufficient; choice of delay (choice of intrinsic time scale) often does not change results much

Problem: unknown D – what are suitable values of m? Depends on analysis method

- Some concepts intentionally use high-dimensional (over-) embedding (e.g. SSA), since they themselves attempt a dimensionality reduction explicitly or implicitly
- Estimation of fractal dimensions: conservative estimates – dimension D of the system (e.g. in the sense of correlation dimension) can only be properly estimated from a time series of length N if $D < log_{10} (N)$
- Consequence: for diagnosing nonlinear characteristic properties, high-dimensional embedding is often not necessary
- Even more: too high-dimensional embedding is often counter-productive – loss of statistical power due to reduced number of state vectors, introduction of numerical artifacts

Chaotic dynamics and exponential divergence

Chaotic systems: trajectories of a <u>deterministic system</u> starting at points that are <u>close in phase space</u> <u>diverge exponentially</u> for a certain time period



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Fractal dimensions

Complex deterministic systems:

- Redundancies between individual variables due to nonlinear interactions
- Scaling behavior of individual components

$\Rightarrow$ non-integer (fractal) dimensions (smaller than topological dimensions of state space)

$\Rightarrow$ transfer of notion from discrete mathematics (fractal sets) to time series analysis

$\Rightarrow$ implies relationships between dynamical characteristics of a time series and geometric characteristics of the trajectory in state space

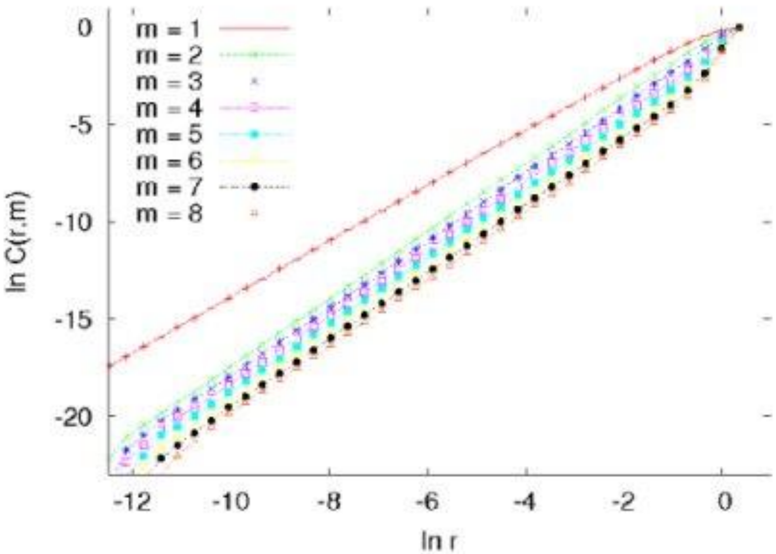
- Fractal dimensions $\Leftrightarrow$ time series roughness ((multi-)fractal formalism)
- Dissipative-chaotic dynamics implies strange attractor (asymptotic set with non-integer dimension)
- Note: non-integer dimensions do <u>not</u> imply chaos (e.g. strange non-chaotic attractors in quasiperiodically driven systems)

Correlation dimension

Correlation integral / correlation sum:

$$C(\varepsilon) = \frac{1}{N^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Theta(\varepsilon - ||\vec{x}(i) - \vec{x}(j)||), \quad \vec{x}(i) \in \mathbb{R}^m$$

$$\boxed{C(\varepsilon, m) \sim \varepsilon^{D_2(m)}}$$



Example: GP algorithm for the Hénon system (source: Scholarpedia)

Scaling behavior:

Grassberger-Procaccia algorithm

=> Requires saturation of scaling with m

Correlation dimension

Practical estimation: remove sojourn points (neighborhood relationships due to temporal auto-correlations – subsequent observations)

$\Rightarrow$ Theiler window

$$\hat{C}(r) = \frac{2}{(N-\tau_c)(N-\tau_c-1)} \sum_{i+\tau_c<j} \theta(r - |x_i - x_j|)$$

Additional requirement: sufficiently long time series (typically: $N \sim 10^D$)

Short and/or noisy time series: no plateau of $\log C / \log \varepsilon$ with increasing embedding dimension m

$\Rightarrow$ Need alternative approaches (e.g., dimension densities)

State space based entropy measures

Approximate entropy

$$ApEn(m,r) = \lim_{N\to\infty}\left[\phi^m(r) - \phi^{m+1}(r)\right] \qquad \phi^m(r) = (N-m+1)^{-1}\sum_{i=1}^{N-m+1}\ln C_i^m(r)$$

Improved measures:

- Sample entropy
- Fuzzy entropy

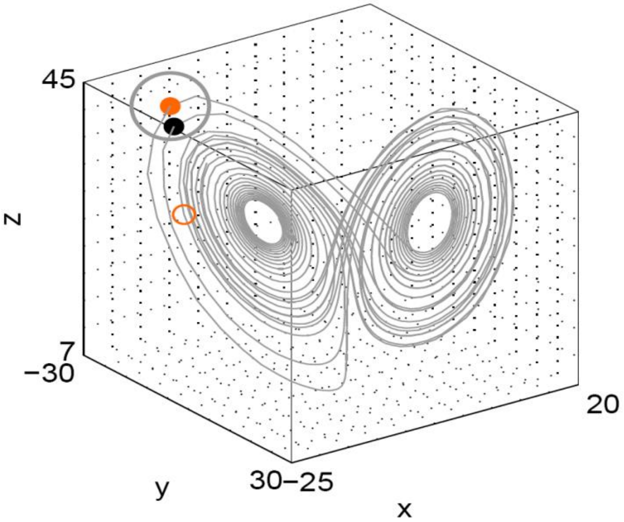
Incorporation of multi-scale information: multi-scale entropy (many algorithmic variants)

Recurrence: General idea

Recurrence of recent states is a typical feature of dynamical systems (Poincaré 1890):

“a system recurs infinitely many times as close as one wishes to its initial state”

Temporal pattern of recurrences encodes fundamental dynamic properties (Robinson & Thiel 2009)



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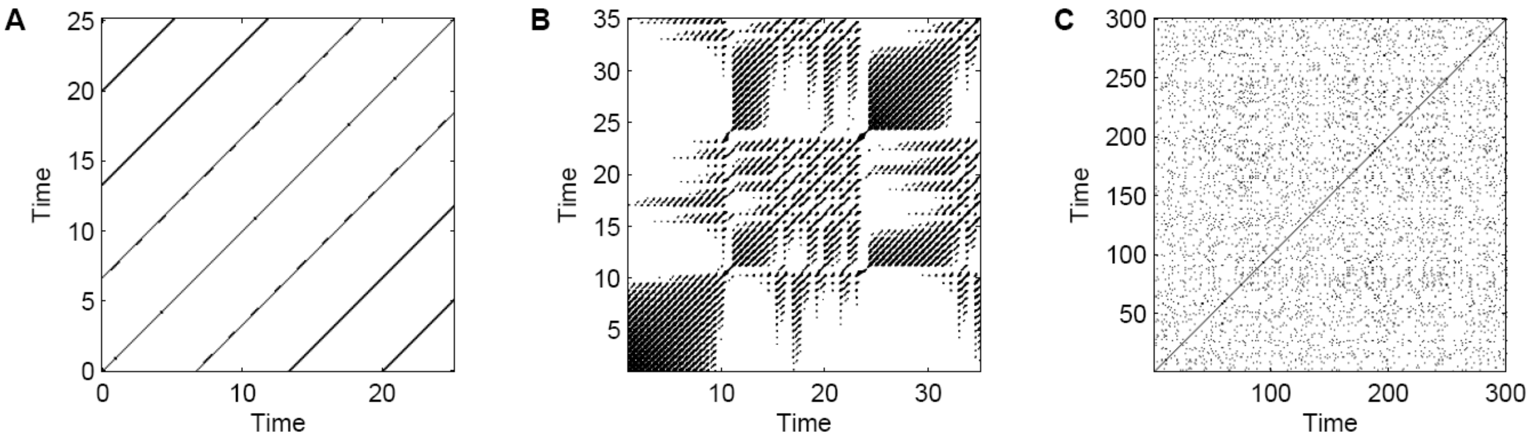
Recurrence plots

Eckmann *et al.* 1987: Visualization of recurrences in terms of "recurrence plots" based on the binary recurrence matrix

$$R_{i,j} = \Theta\left(\varepsilon - d(\vec{x}_i, \vec{x}_j)\right)$$

Periodic **Lorenz (chaotic)** **White noise**



(Donner et al., IJBC, 2011)

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Estimation of dynamical invariants

Diagonal lines indicate predictability – distribution of diagonal line lengths is related to certain dynamical invariants *(Thiel et al., Chaos, 2004)*

$$P_{\varepsilon}^{c}(l) \sim \varepsilon^{D_2} \exp\left(-\hat{K}_2(\varepsilon)\tau l\right)$$

K_2: 2nd-order Rényi entropy (measure for dynamical disorder)
D_2: correlation dimension

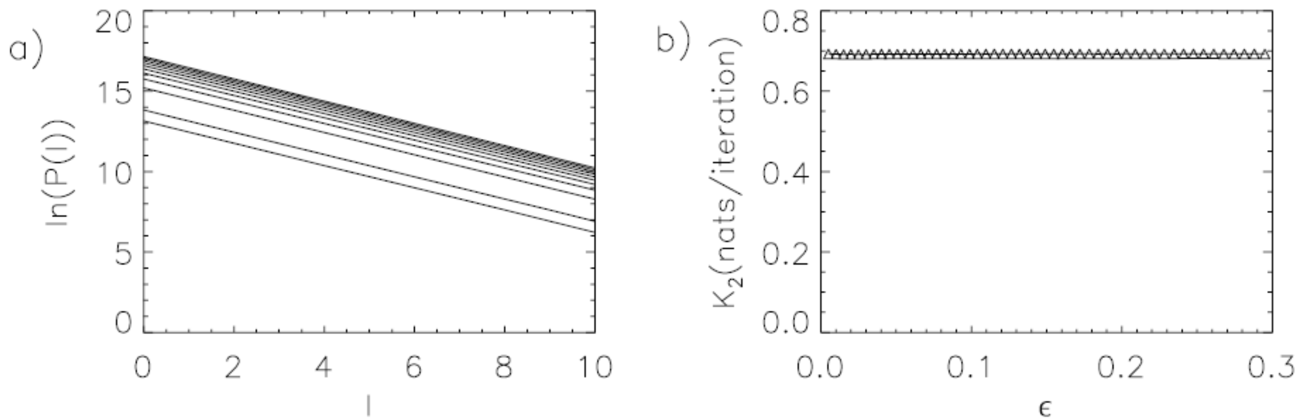


Fig. 2. **(a)** Number of diagonal lines of at least length l versus l in the RP of the Bernoulli map for different values of the threshold ε. The mean slope of the curves is equal to 0.6917. **(b)** Estimator of the Rényi entropy of second order $\hat{K}_2$ vs. ε for the Bernoulli map.

(von Bloh et al., Nonlin. Proc. Geophys., 2005)

Estimation of dynamical invariants

Example: complexity in modeled and observed climate data

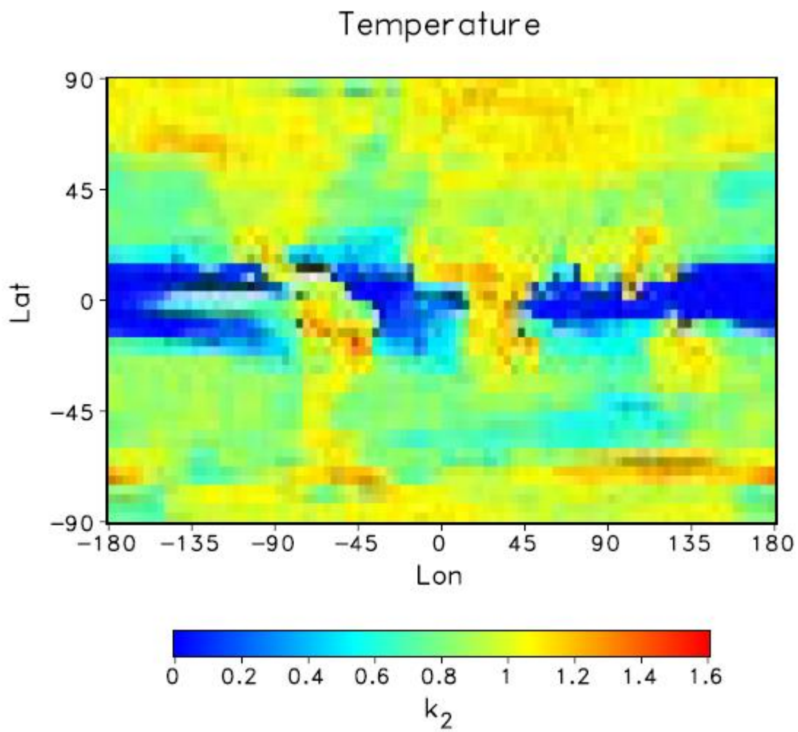


Fig. 5. K_2 estimates for the AOGCM data set for a fixed ε=0.1 K.

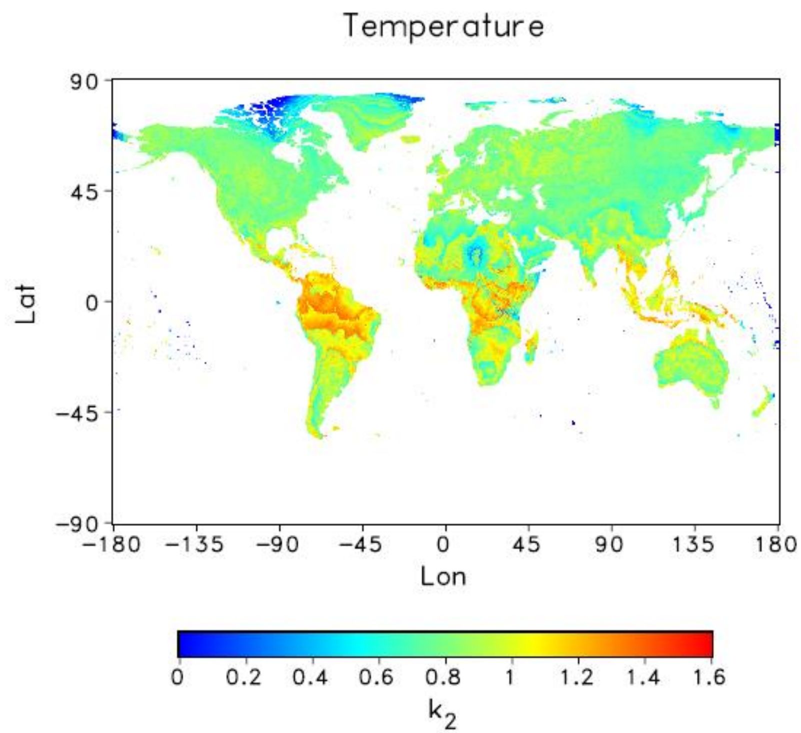


Fig. 7. K_2 estimates for the CRU data set for a fixed ε=0.1 K.

(von Bloh et al., Nonlin. Proc. Geophys., 2005)

Estimation of dynamical invariants

Example: stability of extrasolar planetary systems

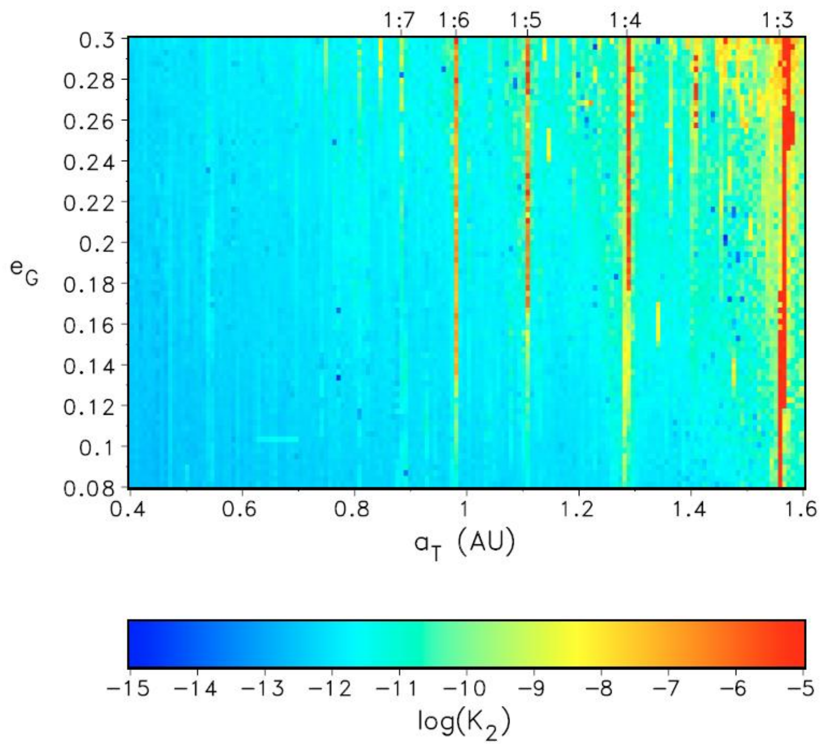


Fig. 11. Entropy plot for HD 72659.

(Asghari et al., A&A, 2004)

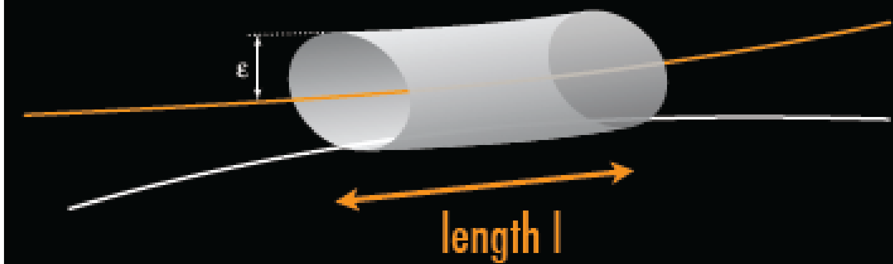
Recurrence quantification analysis

Different types of dynamics result in different appearances of recurrence plots

$\Rightarrow$ quantified in terms of statistical properties of diagonal and vertical "line" length distributions (recurrence quantification analysis) *[Marwan et al., Phys. Rep., 2007]*

$\Rightarrow$ Measures explicitly based on temporal interdependences

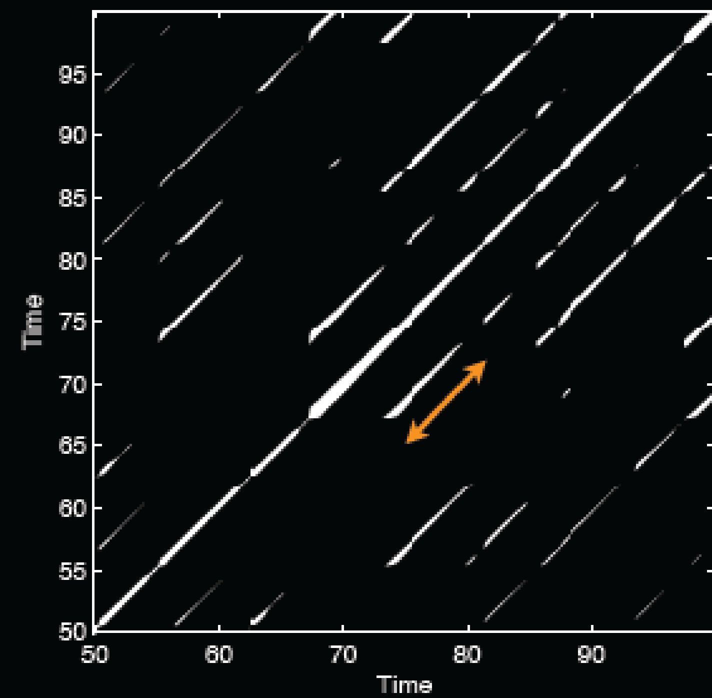
Recurrence quantification analysis



- **Determinism**

$$DET = \frac{\sum_{l=l_{\min}}^{N} l\,P(l)}{\sum_{l=1}^{N} l\,P(l)}$$

Probability that recurrences further recur



Recurrence quantification analysis

Recurrence rate $$\mathrm{RR} = \frac{1}{N^2} \sum_{i,j=1}^{N} \mathbf{R}(i,j)$$

Determinism $$\mathrm{DET} = \frac{\sum_{\ell=\ell_{\min}}^{N} \ell\, P(\ell)}{\sum_{i,j=1}^{N} R(i,j)}$$

Mean diagonal length $$\mathrm{L} = \frac{\sum_{\ell=\ell_{\min}}^{N} \ell\, P(\ell)}{\sum_{\ell=\ell_{\min}}^{N} P(\ell)}$$

Divergence $$DIV = \frac{1}{L_{\max}}$$

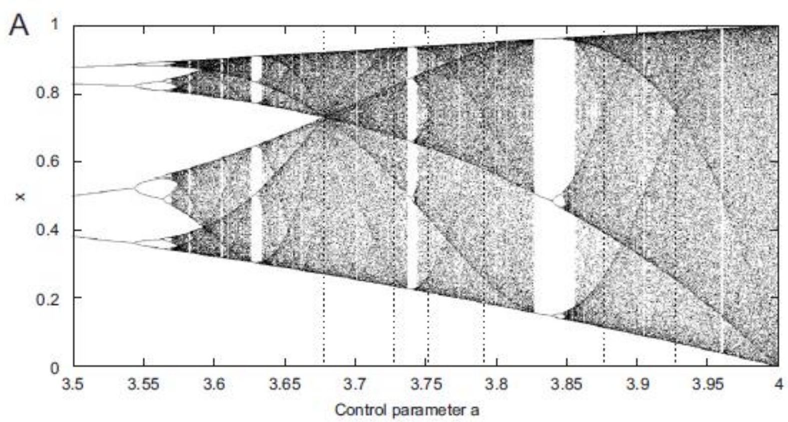
Laminarity $$\mathrm{LAM} = \frac{\sum_{v=v_{\min}}^{N} v P(v)}{\sum_{v=1}^{N} v P(v)}$$

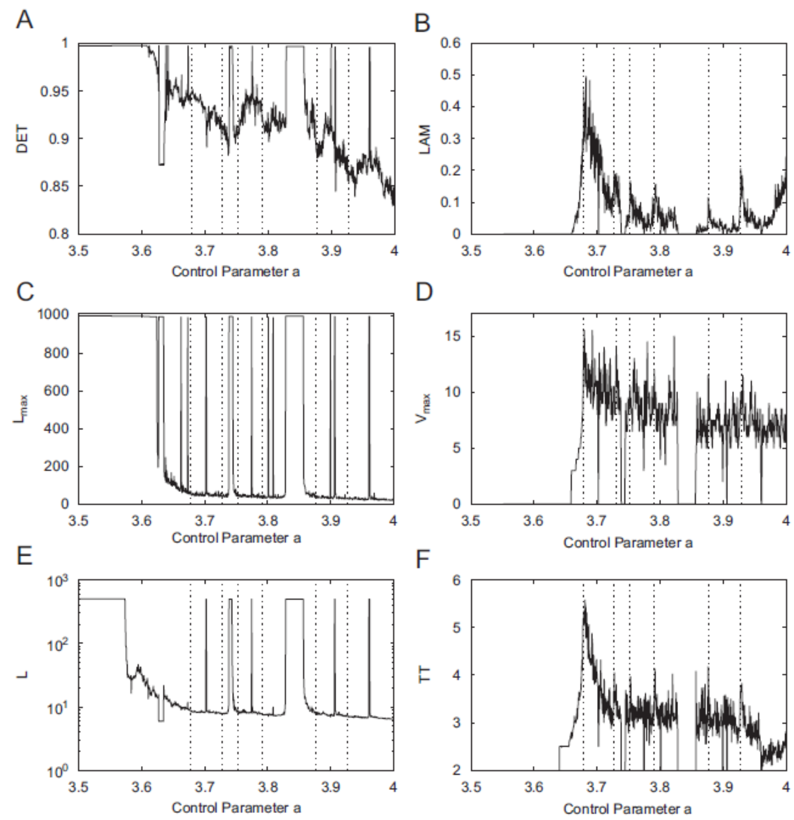
Trapping time $$TT = \frac{\sum_{v=v_{\min}}^{N} v\, P(v)}{\sum_{v=v_{\min}}^{N} P(v)}$$

Recurrence quantification analysis

Statistics on length distributions of diagonal and vertical structures characterize the complexity of dynamics

Example: logistic map

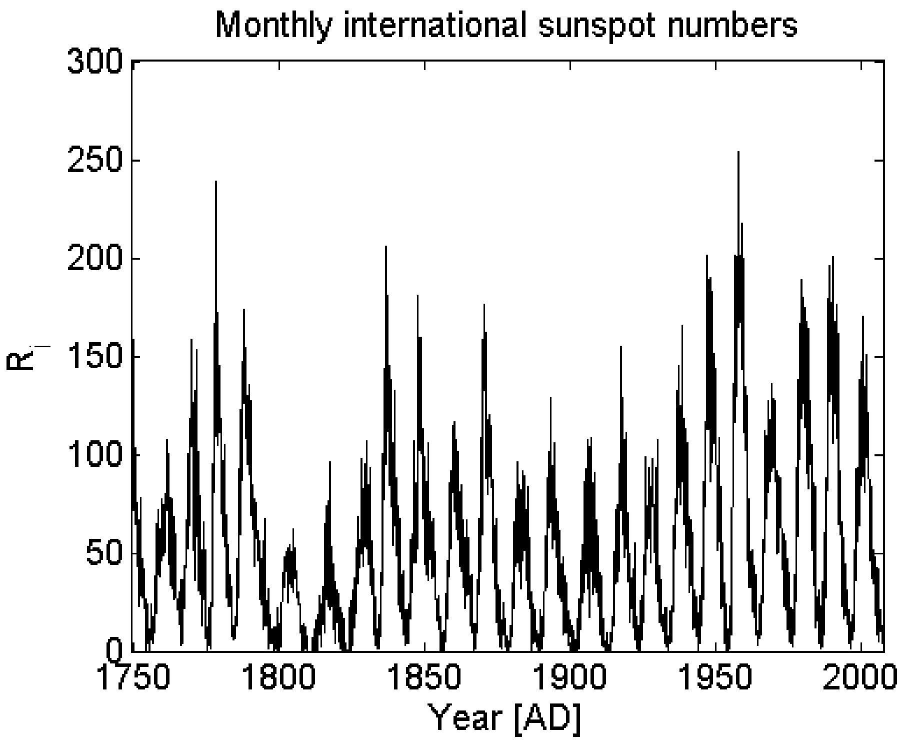


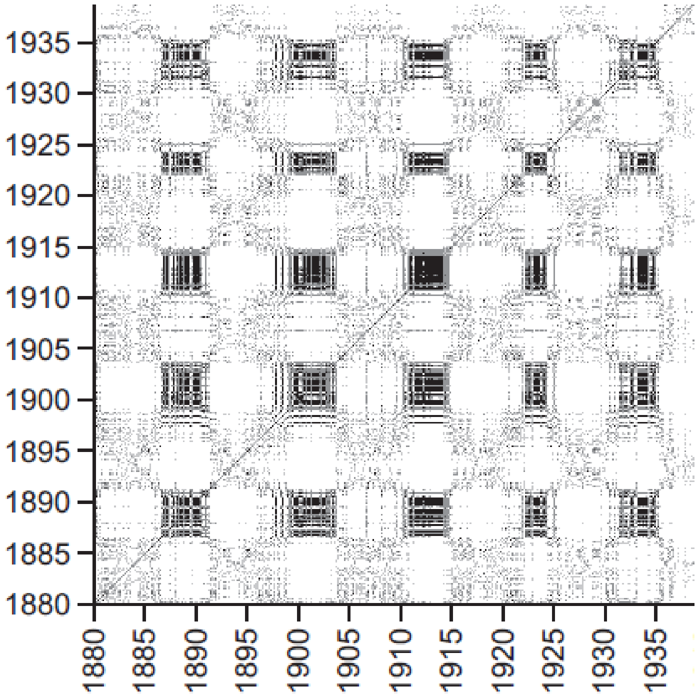


(Marwan et al., Phys. Rep., 2007)

Recurrence quantification analysis

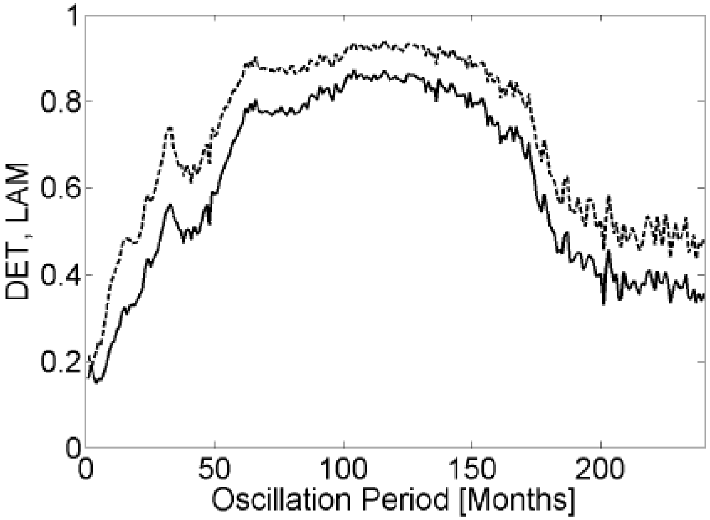
Example: sunspot numbers

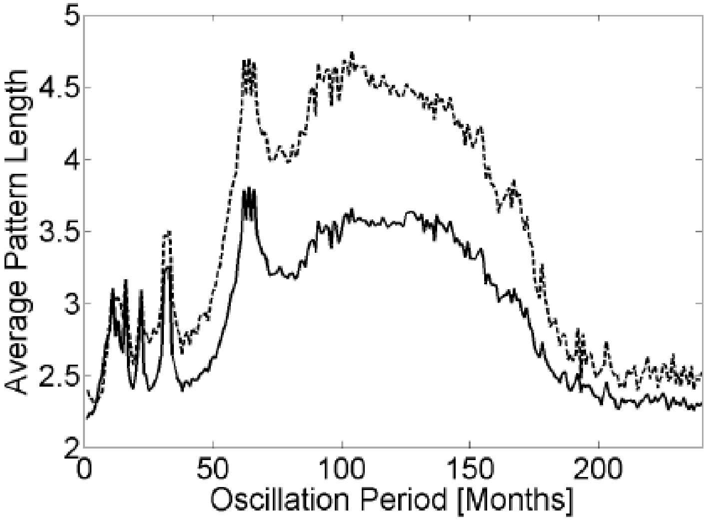




Recurrence quantification analysis

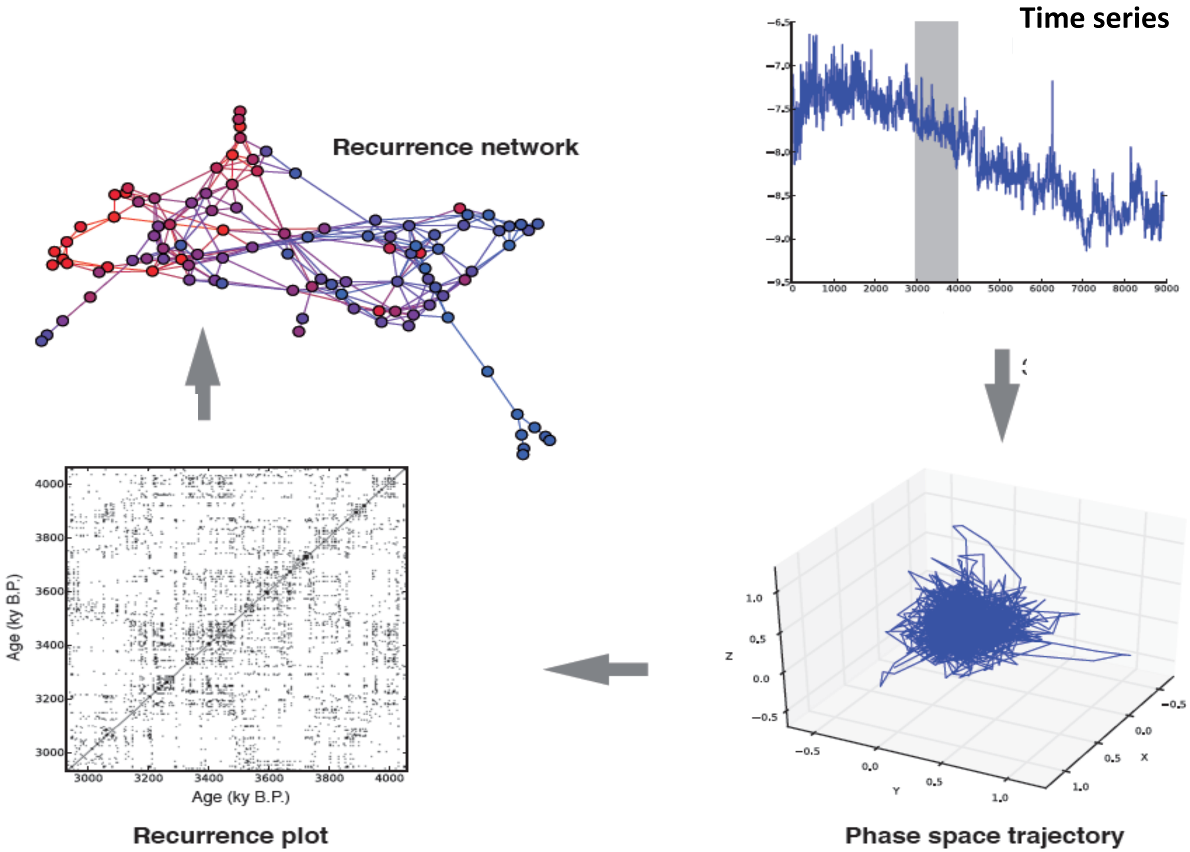
Example: sunspot areas – variability on different scales (time series filtered by continuous wavelet transform with a Morlet wavelet)





(Donner, Lect. Notes Earth Sci., 2008)

Recurrence networks: General idea



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Random geometric graphs

Suppose we have

1. a bound measurable set *S* embedded in a *d*-dimensional metric space with a continuous PDF *p(x)*
2. a set *V* of *N* points drawn randomly according to *p(x)*
3. a function $f: S^2 \rightarrow [0,1]$ with f(x,y)=f(|x-y|) being monotonically decreasing, describing with which probability two elements of *V* at positions *x* and *y* are mutually linked.

An undirected graph *(V, E)* with the elements of the edge set *E* being determined by *f* is called a random geometric graph.

In what follows: $f(x,y) = \Theta(\varepsilon - |x-y|)$

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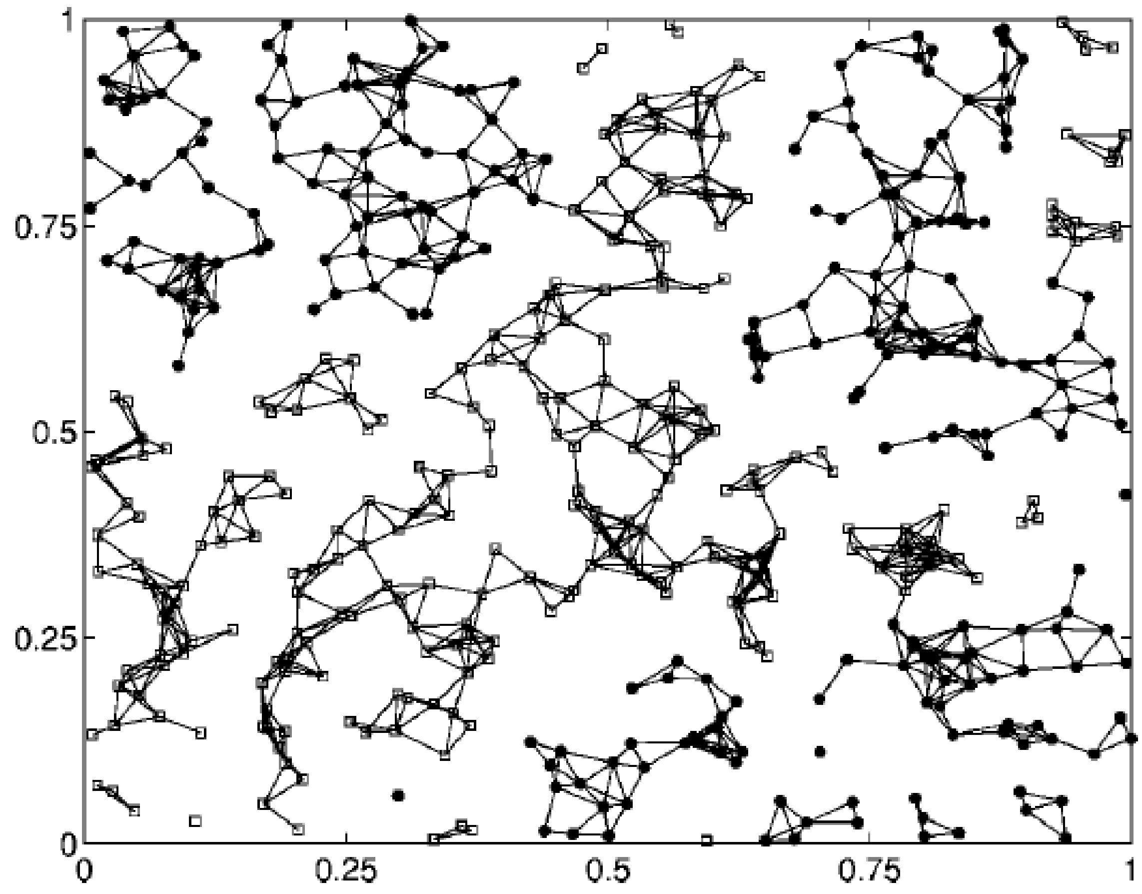
Random geometric graphs



(Dall and Christensen, PRE, 2002)

Random geometric graphs

Why is this relevant here?

Let *S* be the manifold describing a (chaotic) attractor of some (dissipative) dynamical system, *p(x)* being the associated invariant density.

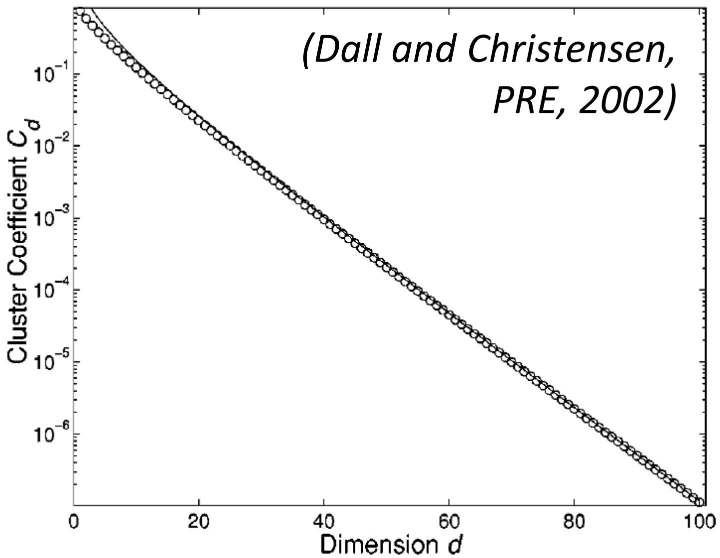
Then the geometric structure of the attractor can be approximated by a finite set of elements of *S* drawn at random according to *p(x)*.

The properties of the resulting random geometric graphs reveal geometric characteristics of the attractor, which are commonly related to dynamical properties.

Random geometric graphs: transitivity

Interesting property describing the attractor geometry: network transitivity (version of the global clustering coefficient)

$$\mathcal{T}(\varepsilon) = \frac{\sum_{i,j,k} A_{ij}(\varepsilon) A_{jk}(\varepsilon) A_{ki}(\varepsilon)}{\sum_{i,j,k} A_{ij}(\varepsilon) A_{ki}(\varepsilon)}$$



Classical result: for a random geometric graph in a metric space of integer dimension d, transitivity scales exponentially with d

Dall and Christensen, PRE, 2002: analytics for Euclidean norm

Donner et al., EPJB, 2011: analytics for maximum norm: $\mathcal{T}=(3/4)^d$

Random geometric graphs: transitivity

What if we construct a random geometric graph for a set *S* that does not fill the d-dimensional space, but has a lower (fractal) dimension?

Conjecture: $\mathcal{T}$ is larger than expected for *d* dimensions, since 3-loops occur more often than for an isotropic alignment of vertices.

Definition of a new notion* of generalized dimension: transitivity dimension

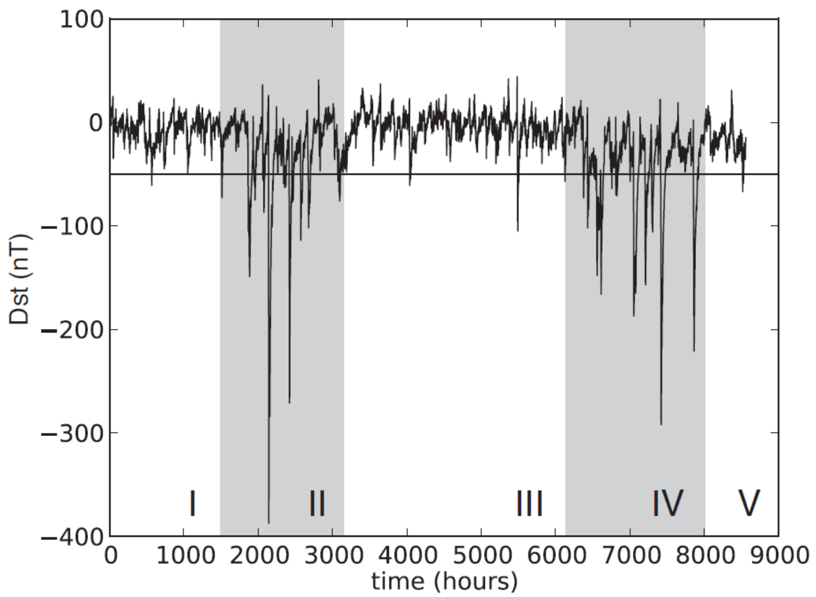
$$D_{\mathcal{T}}(\varepsilon) = \frac{\log \mathcal{T}(\varepsilon)}{\log(3/4)}$$

If *S* homogeneously fills a subset of dimension d: $D_{\mathcal{T}} = d$.

*This notion is different from the classical concept of fractal dimensions based on consideration of scaling characteristics with varying ε.

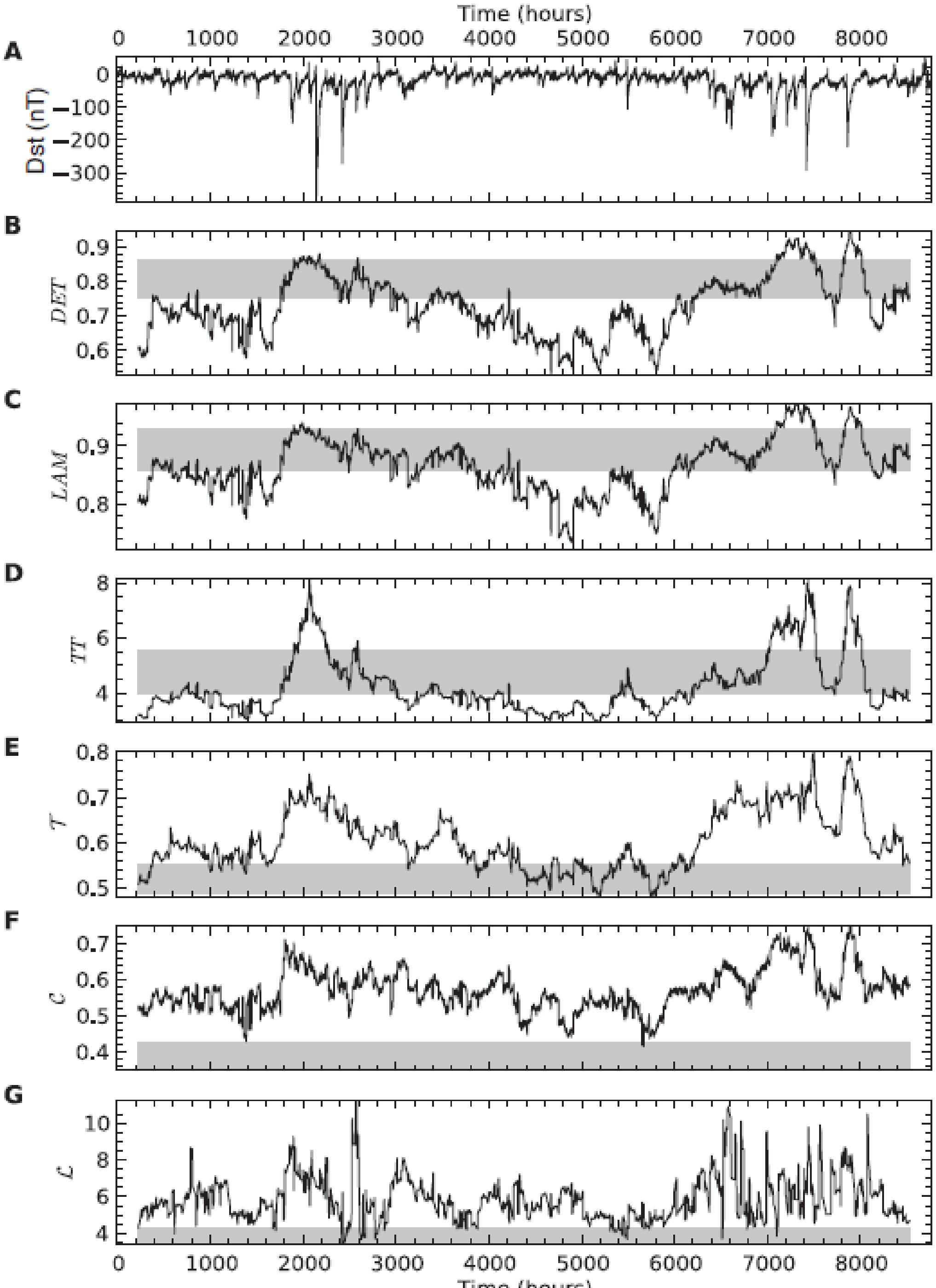
Application: Magnetosphere dynamics

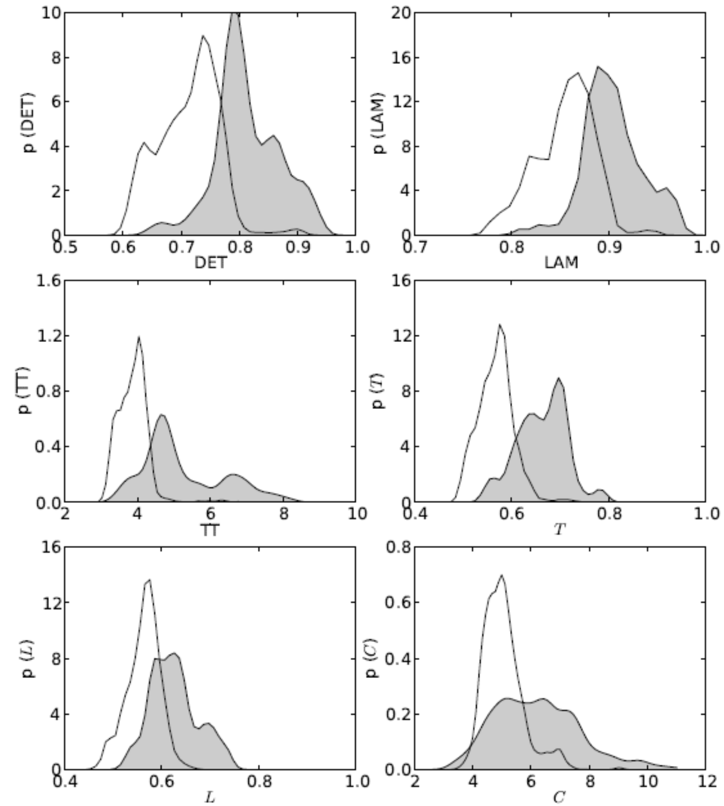
Example: hourly Dst index for year 2001 *(Donner et al., Chaos, 2018)*



Period close to maximum of solar activity cycle
Emergence of two periods with severe geomagnetic storms

Recent work: storm and non-storm periods exhibit distinct temporal organization structure, indicating “pathological” conditions during storms



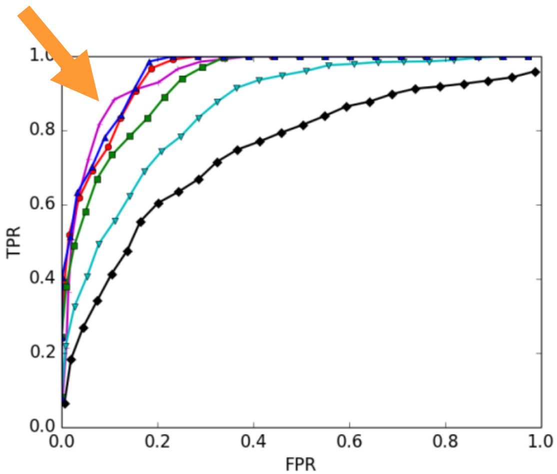


(Donner et al., Chaos, 2018)

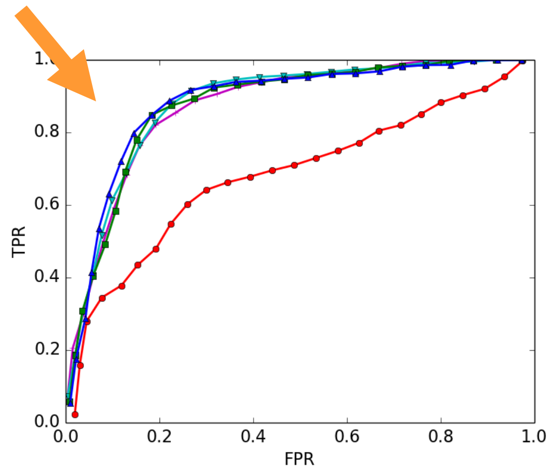
Application: Magnetosphere dynamics

Comparison with other methods (ROC analysis):

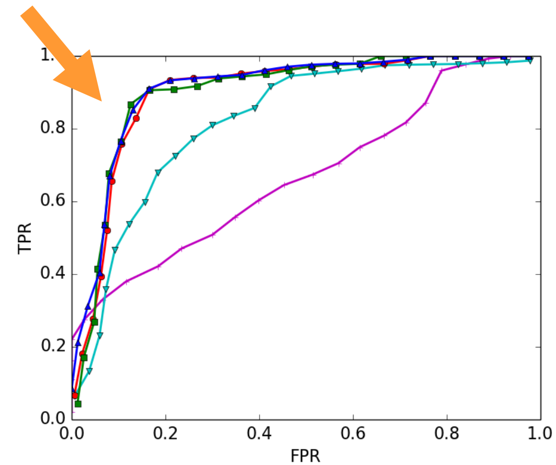
Recurrence-based



Symbolic dynamics based entropies



State space based entropies
Correlation-based measures



(Donner et al., Chaos, 2018)

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Application: Magnetosphere dynamics

Measure	AUC
Shannon entropy	0.670
Block entropy	0.862
Tsallis entropy	0.871
T complexity	0.860
Kolmogorov entropy	0.867
Approximate entropy	**0.893**
Sample entropy	**0.890**
Fuzzy entropy	**0.892**
Hurst exponent	0.730
LVD dimension density	0.845

Best recent approaches also based on phase space viewpoint !

Application: Magnetosphere dynamics

Measure	AUC
Shannon entropy	0.670
Block entropy	0.862
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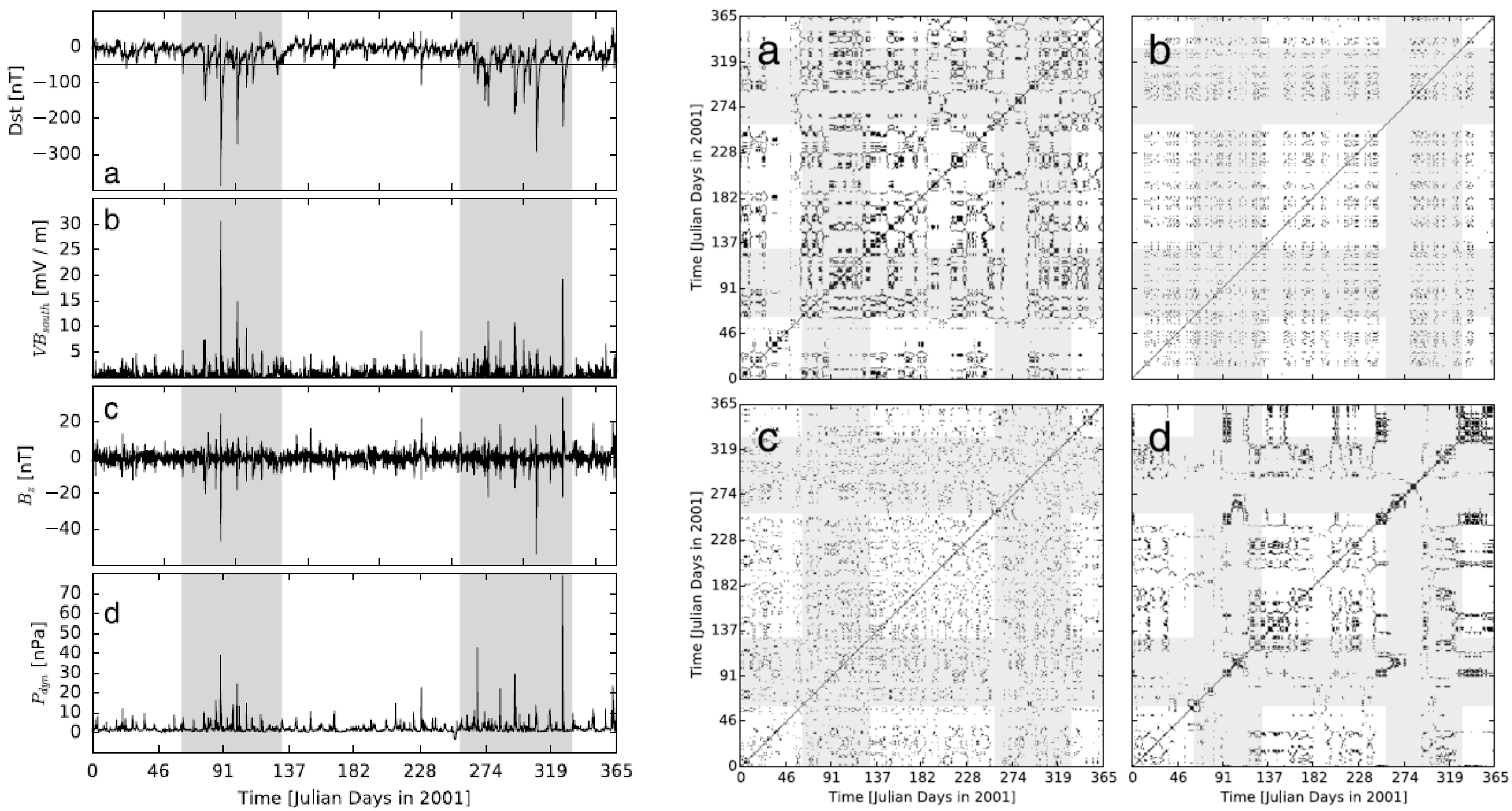
Kolmogorov entropy	0.867
Approximate entropy	**0.893**
Sample entropy	**0.890**
Fuzzy entropy	**0.892**
Hurst exponent	0.730
LVD dimension density	0.845

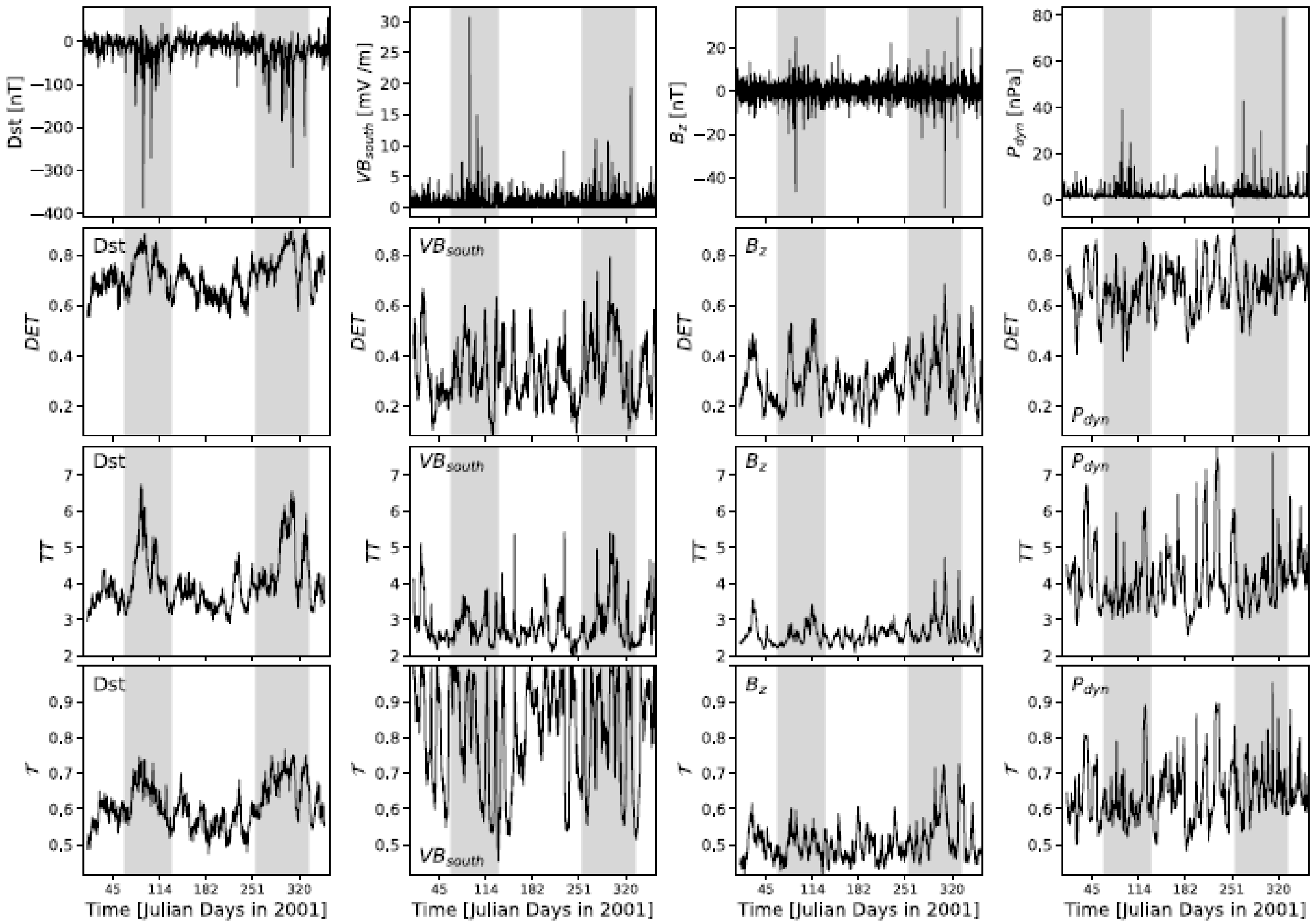
Measure	AUC
Determinism	**0.956**
Laminarity	0.938
Trapping Time	**0.940**
Transitivity	**0.959**
Global Clustering	0.918
Average Path Length	0.806

Extension to solar wind drivers



(Donner et al., JGR Space, 2019)

Extension to solar wind drivers



(Donner et al., JGR Space, 2019)

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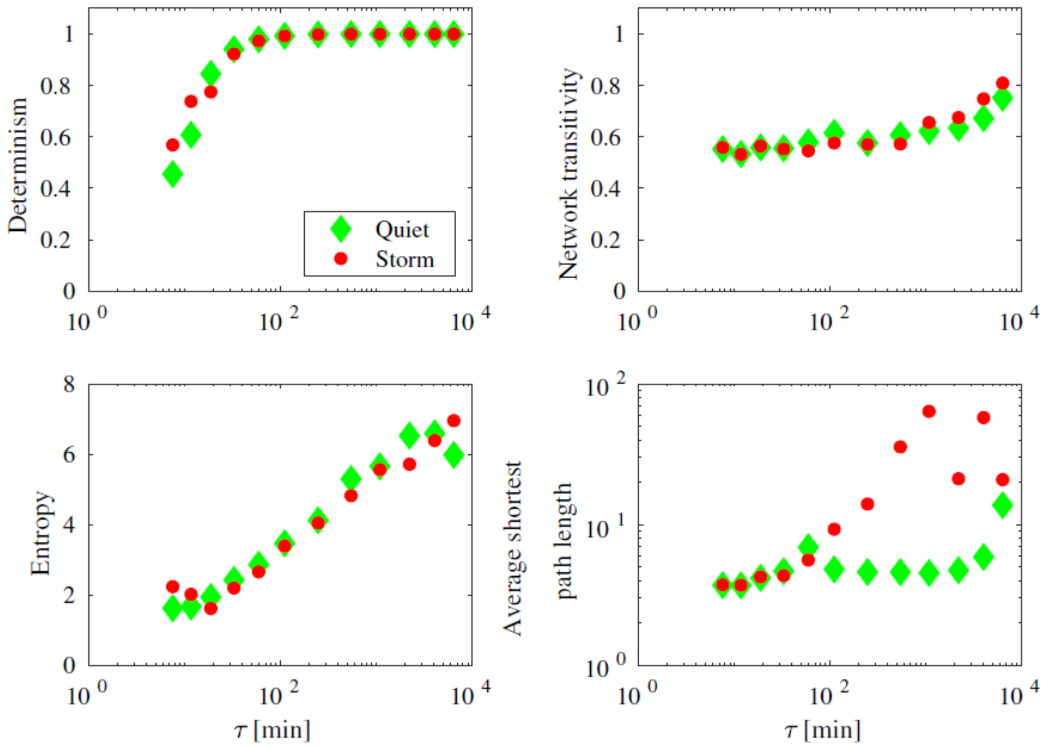
Scale-specific analysis: SYM-H index

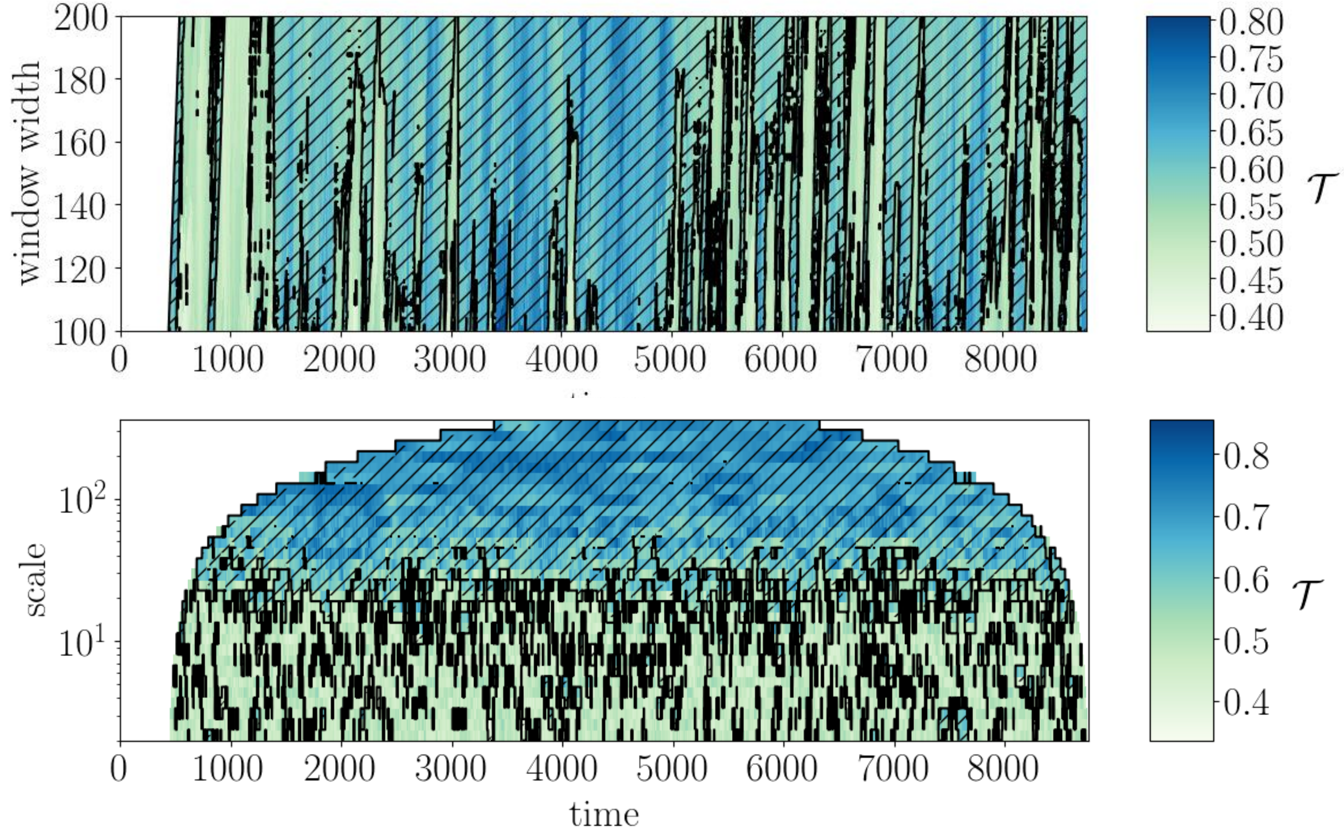
Here: empirical mode decomposition of 1-minute resolution SYM-H time series for two representative time periods in summer 2018, application of recurrence analysis to the different intrinsic mode functions



(Alberti et al., J. Space Weather Space Clim., 2020)

Scale-specific windowed analysis

Here: hourly Dst data for 2015, recurrence network transitivity, pointwise significance testing using shuffle surrogates



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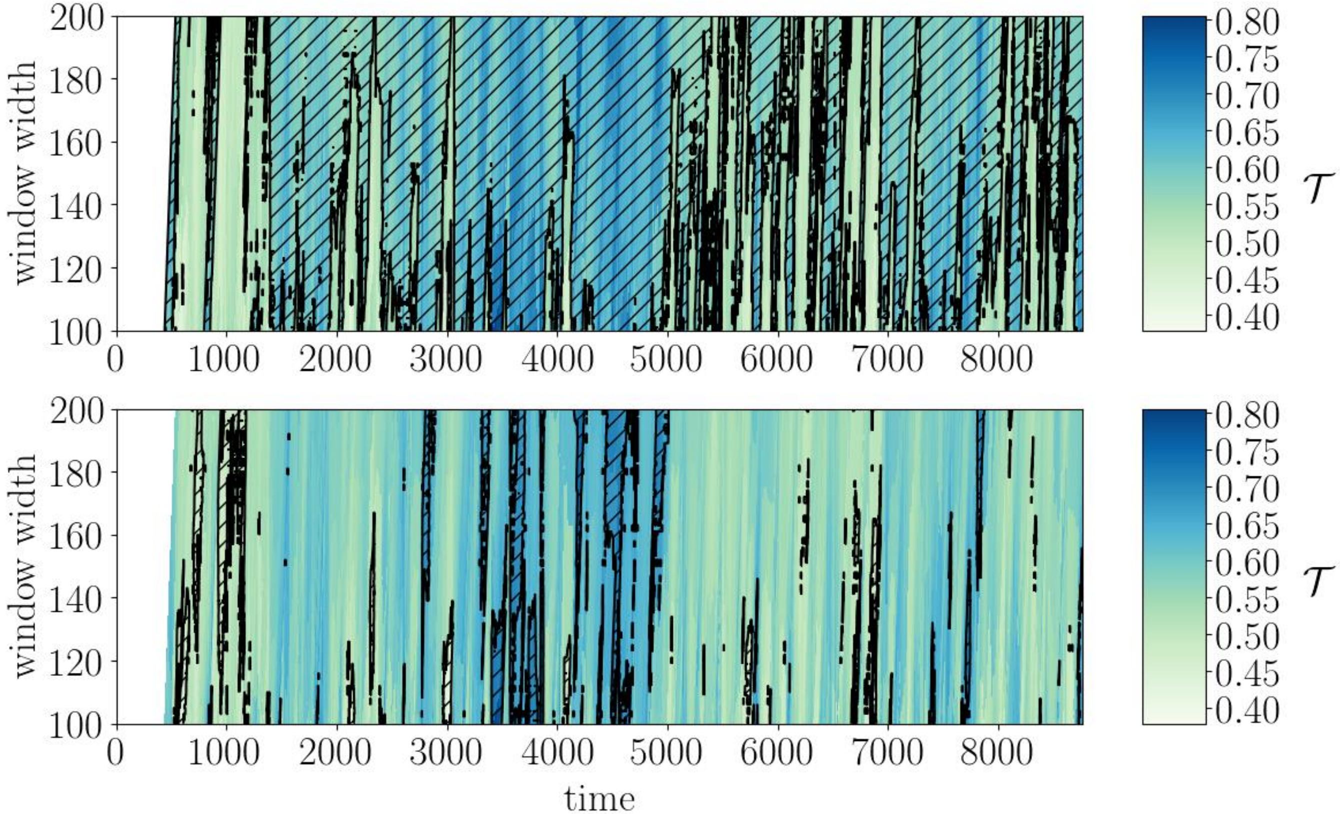
Scale-specific windowed analysis

Comparison: same test with shuffle vs. IAAFT surrogates as benchmarks
(further improvement: areawise significance test of Lekscha and Donner (Proc. R. Soc. A, 2019)



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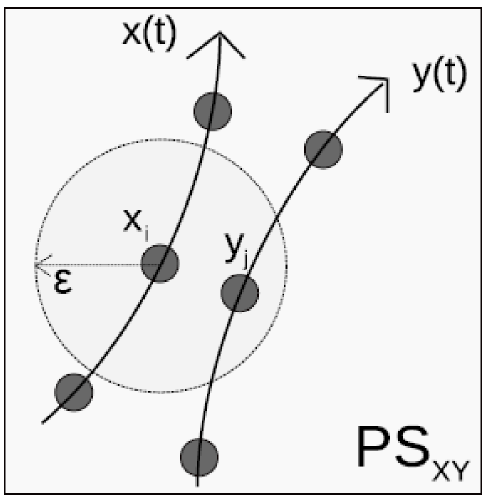
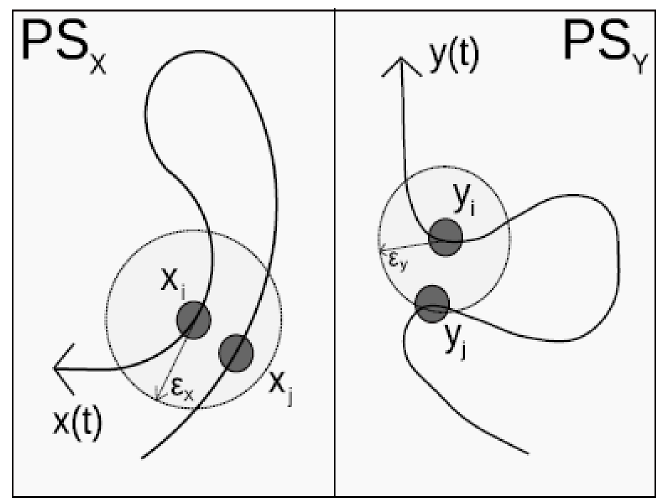
Multivariate extensions of recurrence plots

CROSS RECURRENCE



$$CR_{ij}^{XY}(\varepsilon) = \Theta(\varepsilon - \|x_i - y_j\|)$$

JOINT RECURRENCE



$$JR_{ij}^{XY}(\varepsilon_x, \varepsilon_y) = R_{ij}^{X}(\varepsilon_x) \cdot R_{ij}^{Y}(\varepsilon_y)$$

Multivariate extensions of recurrence plots

Cross recurrence:

$$CR_{ij}^{XY}(\varepsilon) = \Theta(\varepsilon - \|x_i - y_j\|)$$

"Recurrences": Similarities of state vectors

+ Length / sampling not necessarily equal
– Observables must be equal (common phase space, norm)

Joint recurrence:

$$JR_{ij}^{XY}(\varepsilon_x, \varepsilon_y) = R_{ij}^{X}(\varepsilon_x) \cdot R_{ij}^{Y}(\varepsilon_y)$$

Recurrences: Simultaneous recurrences of X and Y

+ Observables not necessarily equal
– Length / sampling must be equal

Cross-recurrence plots

Useful for joint study of two variables representing the same physical quantity and exhibiting the same probability distribution (example: activity indices at both solar hemispheres, velocity/pressure/temperature of a turbulent medium at two points in space,…)

$$
\begin{aligned}
CR_{ij}\left(\varepsilon^{XY}\right) &= CR^{XY}\left(x_i, y_j \mid \varepsilon^{XY}\right) \\
&= \Theta\left(\varepsilon^{XY} - d^{XY}(x_i, y_j)\right)
\end{aligned}
$$

Different analysis options:

1. Cross-recurrence quantification analysis

Note: once the requirements “same observable” and “same PDF” are not met, this analysis can still be performed, yet with questionable interpretability of the results.

Possible modifications: PDF transform of both time series prior to quantitative analysis, “cross-anti-recurrence” for negatively correlated zero-mean quantities (replacing vector difference in distance by vector sum)

Cross-recurrence plots

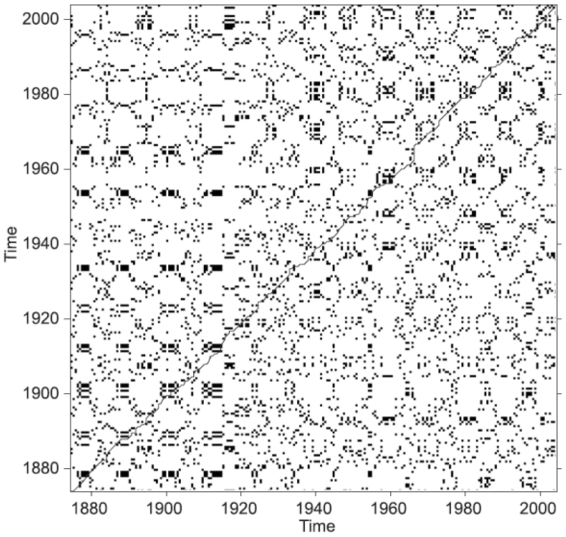
Useful for joint study of two variables representing the same physical quantity and exhibiting the same probability distribution (example: activity indices at both solar hemispheres, velocity/pressure/temperature of a turbulent medium at two points in space,…)

$$CR_{ij}\left(\varepsilon^{XY}\right) = CR^{XY}\left(x_i, y_j \mid \varepsilon^{XY}\right) = \Theta\left(\varepsilon^{XY} - d^{XY}(x_i, y_j)\right)$$

Different analysis options:

1. Cross-recurrence quantification analysis
2. Visual detection of time-dependent synchronization

Example: Zolotova & Ponyavin, A&A, 2006: Northern vs. Southern hemispheric sunspot areas



Applicable to time scale adjustment or dynamic time warping (Marwan et al., NPG, 2002)

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Inter-system recurrence networks

Idea: Combing recurrence and "cross-recurrence" matrices with different densities of intra- and inter-system edges – coupled network paradigm

$$\mathbf{IR}(\boldsymbol{\varepsilon}) = \begin{pmatrix} \mathbf{R}^X(\varepsilon^X) & \mathbf{CR}^{XY}(\varepsilon^{XY}) \\ [\mathbf{CR}^{XY}(\varepsilon^{XY})]^T & \mathbf{R}^Y(\varepsilon^Y) \end{pmatrix}$$

$$\text{with } \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon^X & \varepsilon^{XY} \\ \varepsilon^{XY} & \varepsilon^Y \end{pmatrix}$$

$$\begin{aligned} CR_{ij}(\varepsilon^{XY}) &= CR^{XY}(x_i, y_j \mid \varepsilon^{XY}) \\ &= \Theta(\varepsilon^{XY} - d^{XY}(x_i, y_j)) \end{aligned}$$

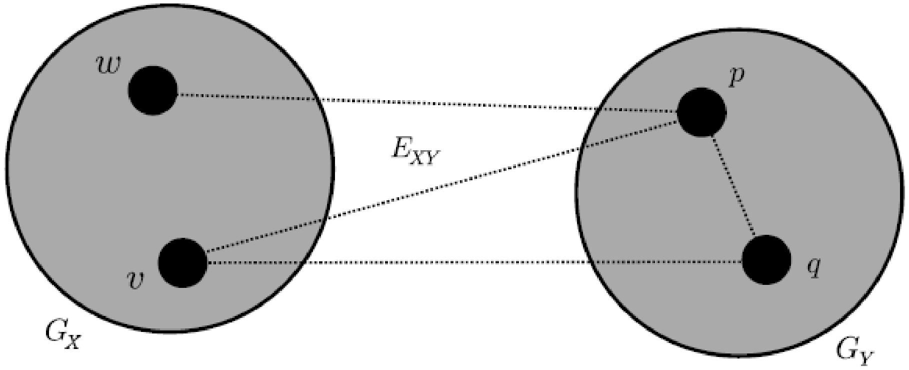


Fig. 2. Two coupled subnetworks. The graph has global cross-clustering coefficients of $\mathcal{C}^{XY} = 0.5 \neq \mathcal{C}^{YX} = 0$ and cross-transitivities $\mathcal{T}^{XY} = 1 \neq \mathcal{T}^{YX} = 0$.

Inter-system recurrence networks

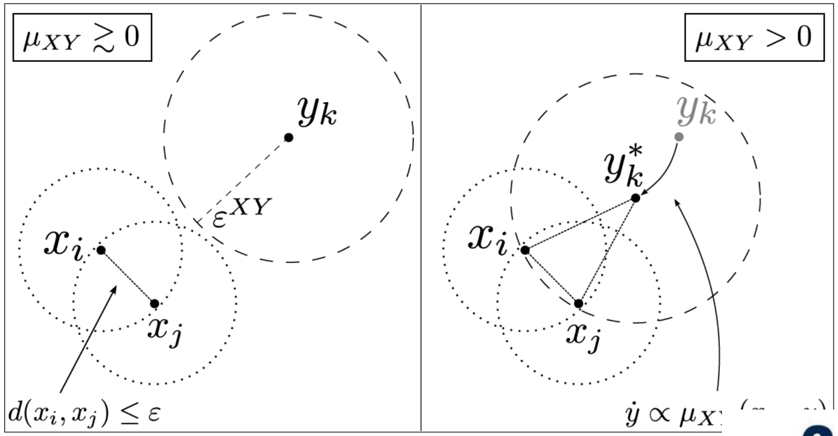
Asymmetries between interacting network measures: coupling direction

Expected qualitative behaviour of IRN measures in different coupling situations for systems with comparable properties in the absence of (generalised) synchronisation.

Coupling direction	Expected relation in network measures
no coupling	$\mathcal{T}^{XY} \approx \mathcal{T}^{YX}$, $\mathcal{C}^{XY} \approx \mathcal{C}^{YX}$
$X \to Y$	$\mathcal{T}^{XY} < \mathcal{T}^{YX}$, $\mathcal{C}^{XY} < \mathcal{C}^{YX}$
$Y \to X$	$\mathcal{T}^{XY} > \mathcal{T}^{YX}$, $\mathcal{C}^{XY} > \mathcal{C}^{YX}$
$X \leftrightarrow Y$	$\mathcal{T}^{XY} \approx \mathcal{T}^{YX}$, $\mathcal{C}^{XY} \approx \mathcal{C}^{YX}$

so far only heuristic arguments!

(Feldhoff et al., Phys. Lett. A, 2012)



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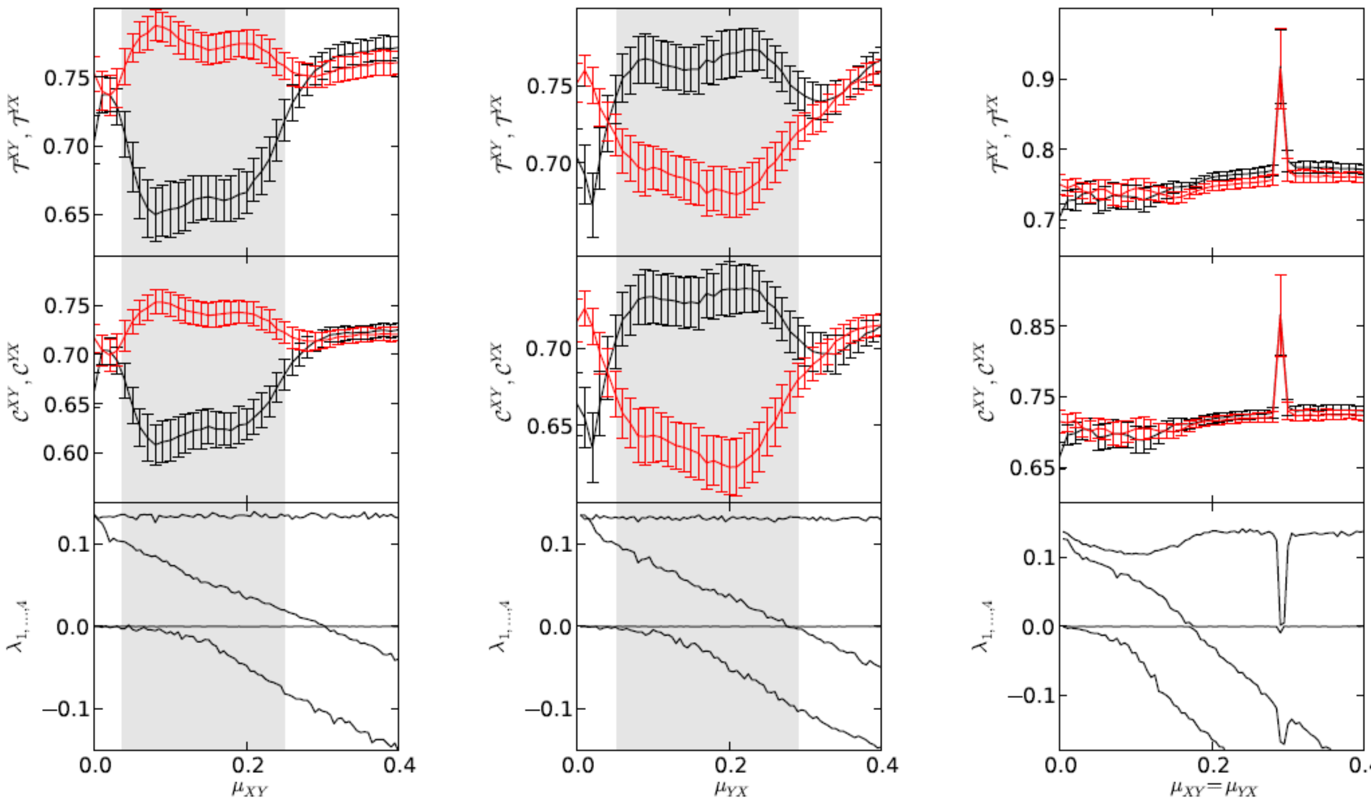
Inter-system recurrence networks

Example: Two diffusively coupled Rössler oscillators

(Feldhoff et al., Phys. Lett. A, 2012)

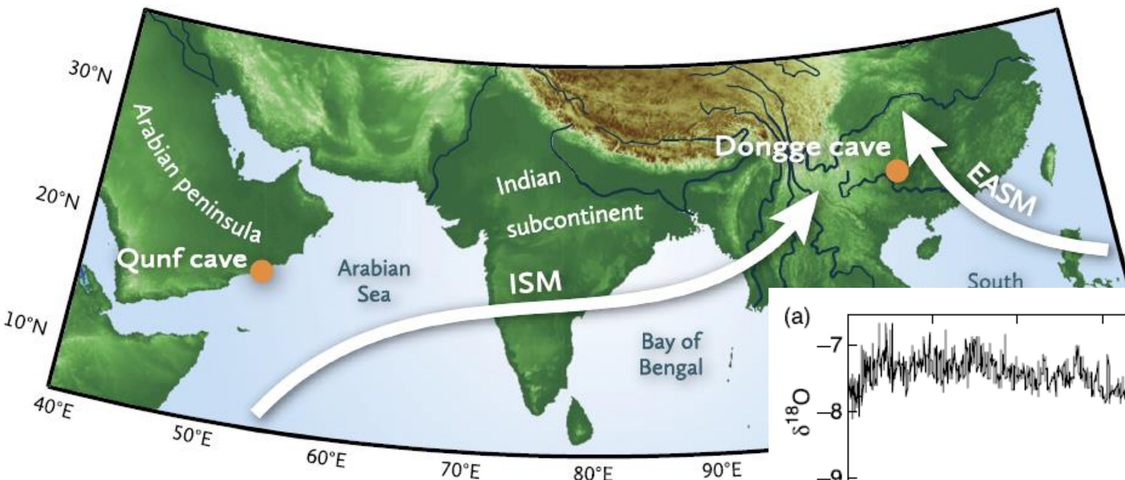


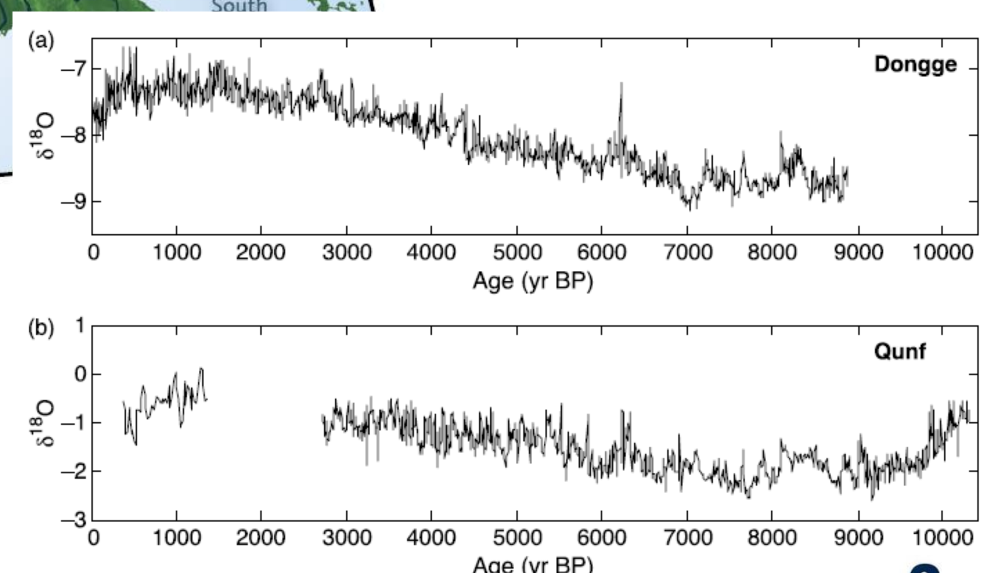
(a) Coupling: $X \to Y$ (b) Coupling: $Y \to X$ (c) Coupling: $X \leftrightarrow Y$

Inter-system recurrence networks

Example: Coupling between Indian and East Asian Summer Monsoon

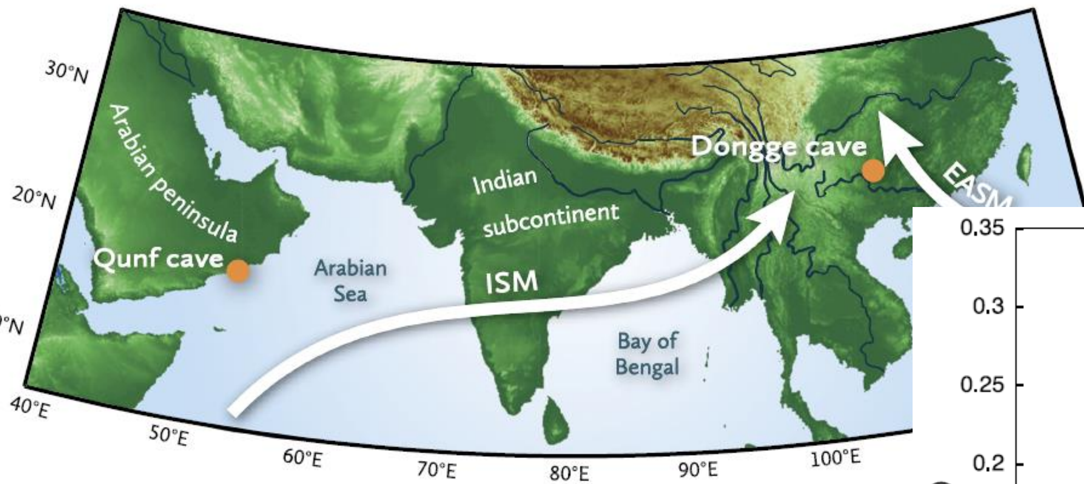
(Feldhoff et al., Phys. Lett. A, 2012)



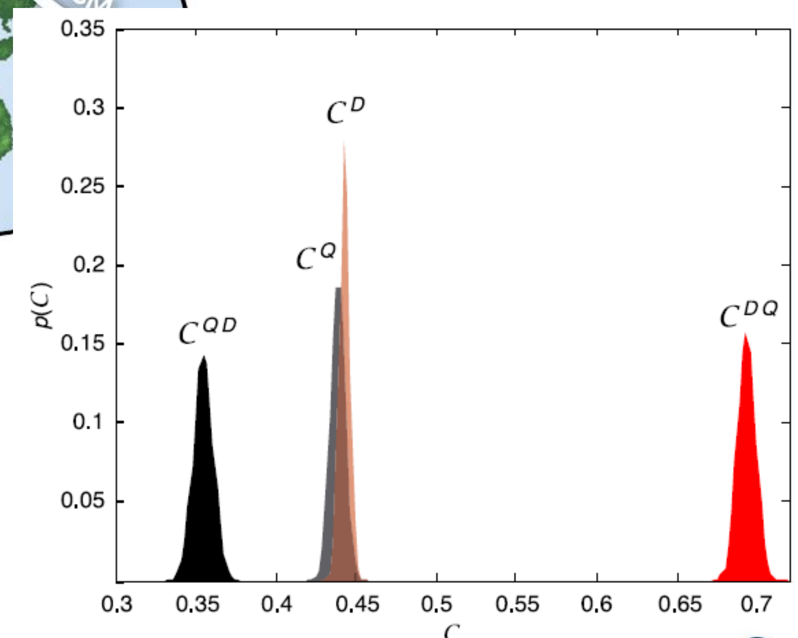


Inter-system recurrence networks

Example: Coupling between Indian and East Asian Summer Monsoon



(Feldhoff et al., Phys. Lett. A, 2012)



Result: clear indication for a coupling ISM => EASM

Joint recurrences

Idea: study simultaneous recurrences of two or more systems (Romano et al., EPL, 2005; Feldhoff et al., EPL, 2013)

$$JR_{ij}(\varepsilon_x, \varepsilon_y) = \Theta(\varepsilon_x - \|x_i - x_j\|)\Theta(\varepsilon_y - \|y_i - y_j\|)$$
$$A_{ij}(\varepsilon_x, \varepsilon_y) = JR_{ij}(\varepsilon_x, \varepsilon_y) - \delta_{ij}$$

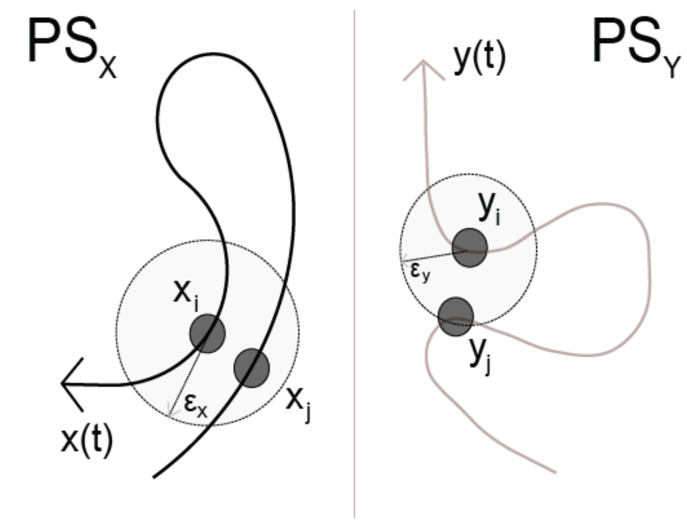


Figure 1: Illustration of a joint recurrence of systems X and Y.

(Feldhoff et al., EPL, 2013)

Joint recurrences & Generalized synchronization

Allows for general functional relationship between different systems (Rulkov et al., 1995) – detection is problematic

Romano et al., EPL, 2005: in presence of GS, recurrences of both systems occur at the same time – recurrence plots become the same, density of points in joint RP approaches that of single-system recurrence plots

$$RR^{\boldsymbol{x}} = \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(\varepsilon_{\boldsymbol{x}} - ||\boldsymbol{x}_i - \boldsymbol{x}_j||)$$

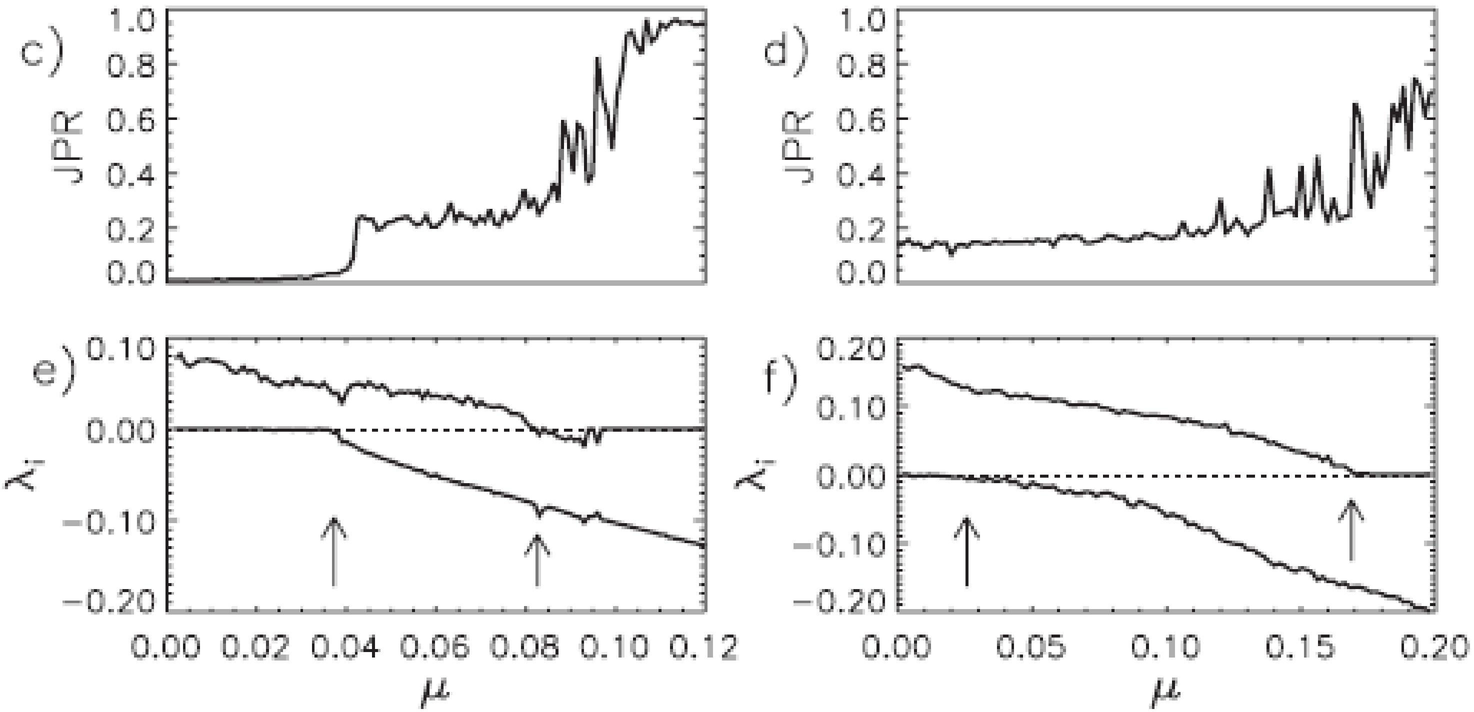
$$RR^{\boldsymbol{x},\boldsymbol{y}} = \frac{1}{N^2} \sum_{i,j=1}^{N} \Theta(\varepsilon_{\boldsymbol{x}} - ||\boldsymbol{x}_i - \boldsymbol{x}_j||)\Theta(\varepsilon_{\boldsymbol{y}} - ||\boldsymbol{y}_i - \boldsymbol{y}_j||)$$

$$S(\tau) = \frac{\frac{1}{N^2} \sum_{i,j}^{N} \Theta(\varepsilon_{\boldsymbol{x}}^i - ||\boldsymbol{x}_i - \boldsymbol{x}_j||)\Theta(\varepsilon_{\boldsymbol{y}}^i - ||\boldsymbol{y}_{i+\tau} - \boldsymbol{y}_{j+\tau}||)}{RR}$$

$$\boxed{JPR = \max_{\tau} \frac{S(\tau) - RR}{1 - RR}}$$

Joint recurrences & Generalized synchronization

Example: two coupled Rössler oscillators *(Romano et al., EPL, 2005)*



Joint recurrences & Generalized synchronization

Problem: limit of *JPR=1* is hardly approached in complex scenarios (e.g., coupled Rössler systems in funnel regime)

Idea: use higher-order characteristics (three-point relations: transitivity)

absence of synchronization:

$$\mathcal{T}_X, \mathcal{T}_Y \gg \mathcal{T}_J$$

$$D_{\mathcal{T}_J} = \log \mathcal{T}_J / \log(3/4) \approx D_{\mathcal{T}_X} + D_{\mathcal{T}_Y}$$

generalized synchronization: locking of effective dynamical degrees of freedom

$$\mathcal{T}_J \rightarrow \mathcal{T}_X, \mathcal{T}_Y$$

Characteristic parameter: transitivity ratio

$$\boxed{Q_{\mathcal{T}} = \frac{\mathcal{T}_J}{(\mathcal{T}_X + \mathcal{T}_Y)/2}}$$

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Example: Coupled Rössler systems

$$\dot{x_1} = -(1+\nu)x_2 - x_3$$
$$\dot{x_2} = (1+\nu)x_1 + ax_2 + \mu_{YX}(y_2 - x_2)$$
$$\dot{x_3} = b + x_3(x_1 - c)$$
$$\dot{y_1} = -(1-\nu)y_2 - y_3$$
$$\dot{y_2} = (1-\nu)y_1 + ay_2 + \mu_{XY}(x_2 - y_2)$$
$$\dot{y_3} = b + y_3(y_1 - c)$$

Example: Coupled Rössler systems

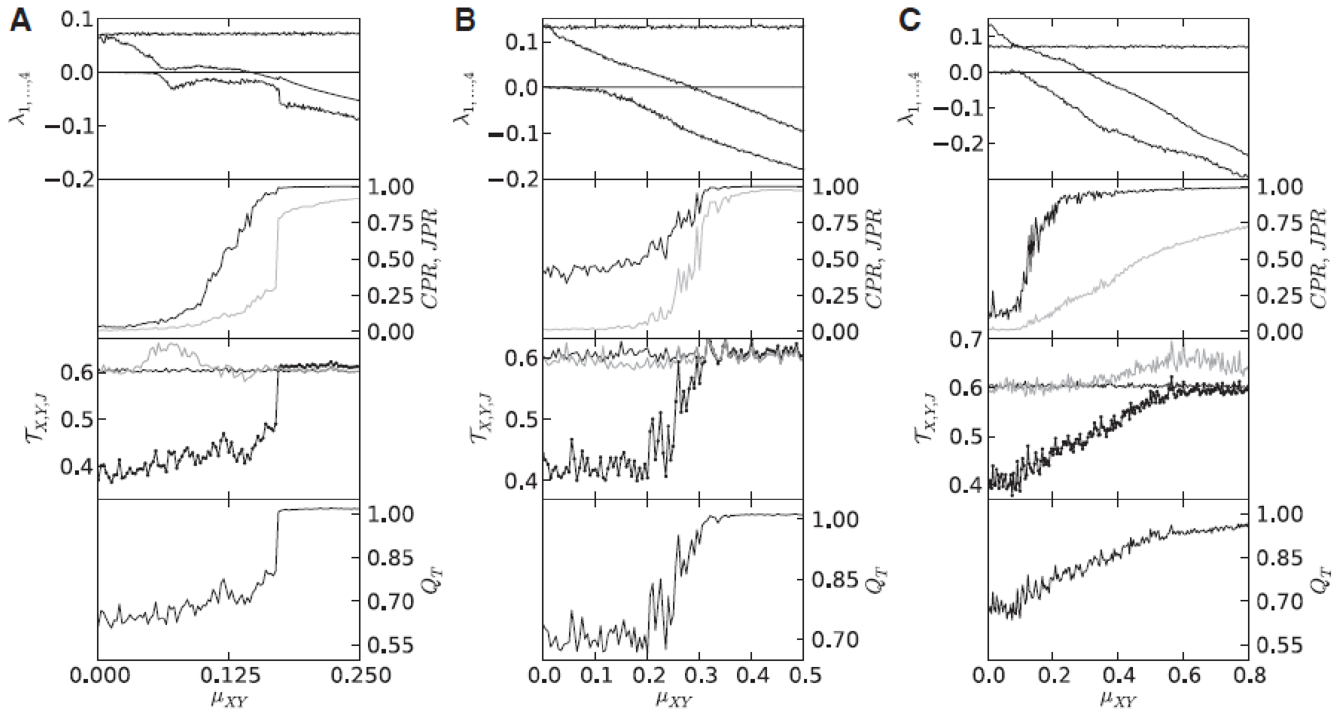


Figure 2: Results of synchronisation analysis for two unidirectionally ($X \to Y$) coupled Rössler systems being (A) both in the phase-coherent regime, (B) both in the funnel regime, and (C) in phase-coherent (X) and funnel regime (Y): the four largest Lyapunov exponents $\lambda_1, \ldots, \lambda_4$ estimated using the Wolf algorithm [46] (using $N = 9,000,000$ data points from simulations with step size $h = 0.001$ starting at $T = 1,000$); recurrence-based synchronisation indices CPR (black) and JPR (grey) [17]; transitivities of individual and joint recurrence networks $\mathcal{T}_X$ (dark grey), $\mathcal{T}_Y$ (light grey) and $\mathcal{T}_J$ (black); transitivity ratio $Q_{\mathcal{T}}$ (from top to bottom).

(Feldhoff et al., EPL, 2013)

Example: Coupled Rössler systems

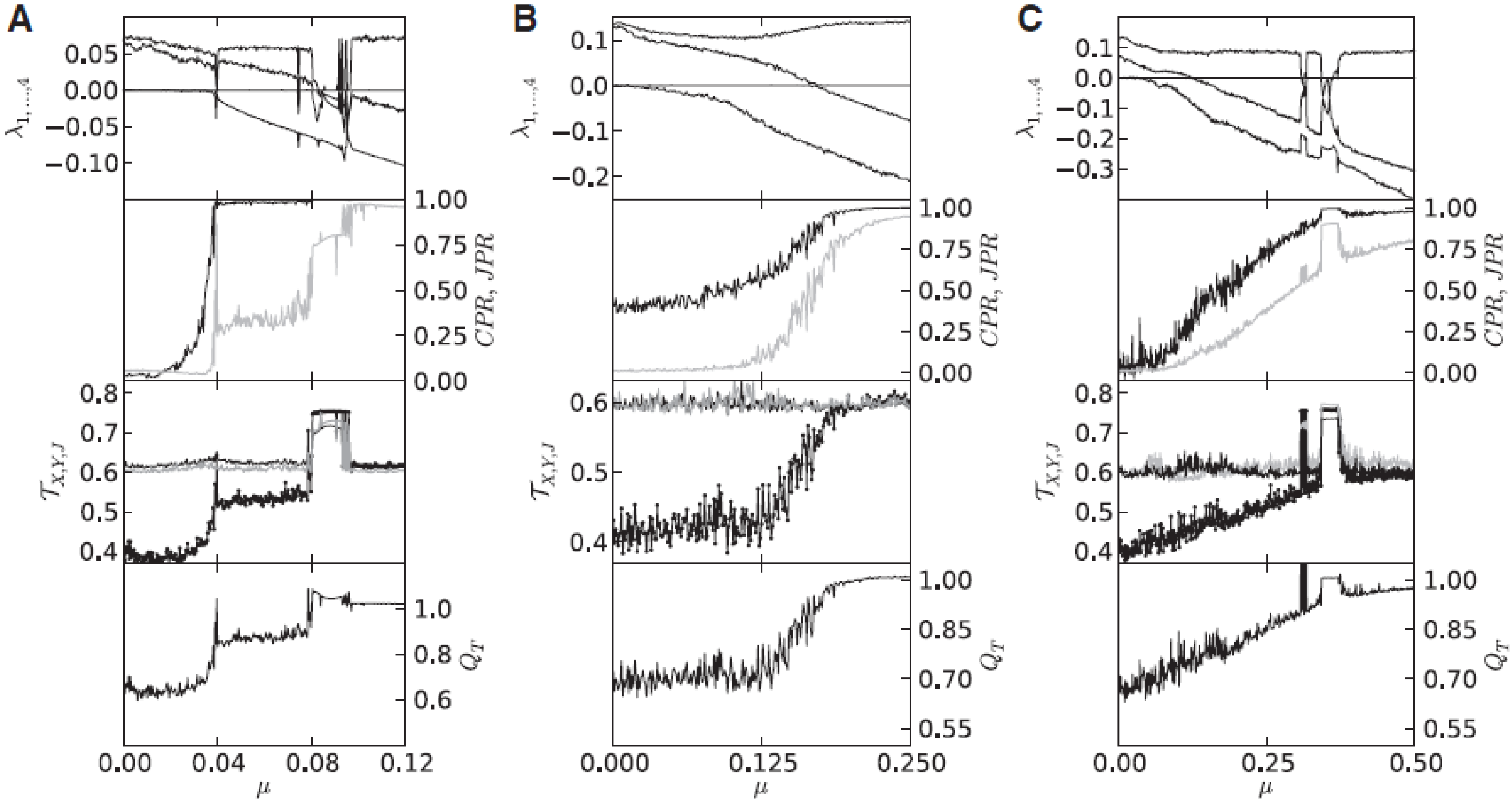


Figure 3: As in Fig. 2 for bidirectional coupling.

(*Feldhoff et al., EPL, 2013*)

Conditional recurrence

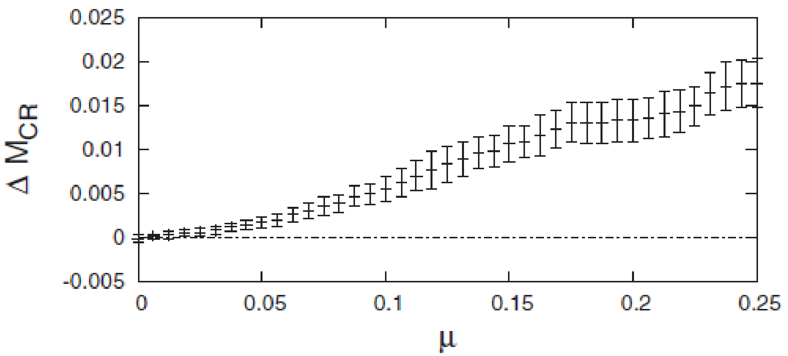
Coupling direction between two variables can be estimated using conditional recurrence probabilities (Romano et al., PRE, 2007)

$$M_{CR}(Y|X) = \frac{1}{N}\sum_{i=1}^{N} p(\vec{y}_i|\vec{x}_i) = \frac{1}{N}\sum_{i=1}^{N} \frac{\sum_{j=1}^{N} J_{R\,i,j}^{X,Y}}{\sum_{j=1}^{N} R_{i,j}^{X}}$$

$$M_{CR}(X|Y) = \frac{1}{N}\sum_{i=1}^{N} p(\vec{x}_i|\vec{y}_i) = \frac{1}{N}\sum_{i=1}^{N} \frac{\sum_{j=1}^{N} J_{R\,i,j}^{X,Y}}{\sum_{j=1}^{N} R_{i,j}^{Y}}$$

$$\text{if } X \text{ drives } Y, \quad M_{CR}(Y|X) < M_{CR}(X|Y)$$

$$\text{if } Y \text{ drives } X, \quad M_{CR}(X|Y) < M_{CR}(Y|X)$$



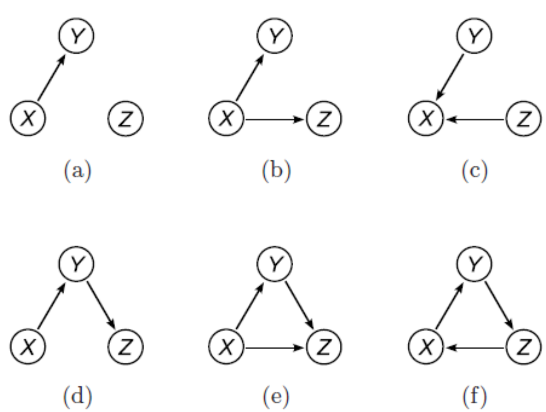
Example: chaotic Rössler system driven by a stochastic van-der-Pol system (Romano et al., PRE, 2007)

Conditional recurrence

Partial mean conditional recurrence probability (Zou et al., IJBC, 2011: distinguish direct from indirect couplings if more than two variables are involved

$$\Delta \mathrm{MCR}(Y \mid X)_Z = -(\mathrm{MCR}(Y \mid X) - \mathrm{MCR}(Y \mid X, Z))$$

$$\Delta \mathrm{MCR}(X \mid Y)_Z = -(\mathrm{MCR}(X \mid Y) - \mathrm{MCR}(X \mid Y, Z))$$



	Univariate: (RR_X, RR_Y, RR_Z)		Bivariate: $(\Delta MCR(X \mid Y), \Delta MCR(X \mid Z), \Delta MCR(Y \mid Z))$		Partial MCR	
(a) (b) (c) (d) (e) (f)	(a), (b): $(+, -, -)$	(a)	$(+, -, -)$	(a)		(a)
		(c)	$(+, +, -)$	(c)		(c)
	(c), (d), (e): $(+, +, -)$	(b)	$(+, +, -)$	(b)		(b)
		(d)	$(+, +, +)$	(d)	$MCR(Z \mid Y) > MCR(Z \mid X)$	(d)
		(e)		(e)	$MCR(Z \mid Y) < MCR(Z \mid X)$	(e)
	(f): $(+, +, +)$	(f)				(f)

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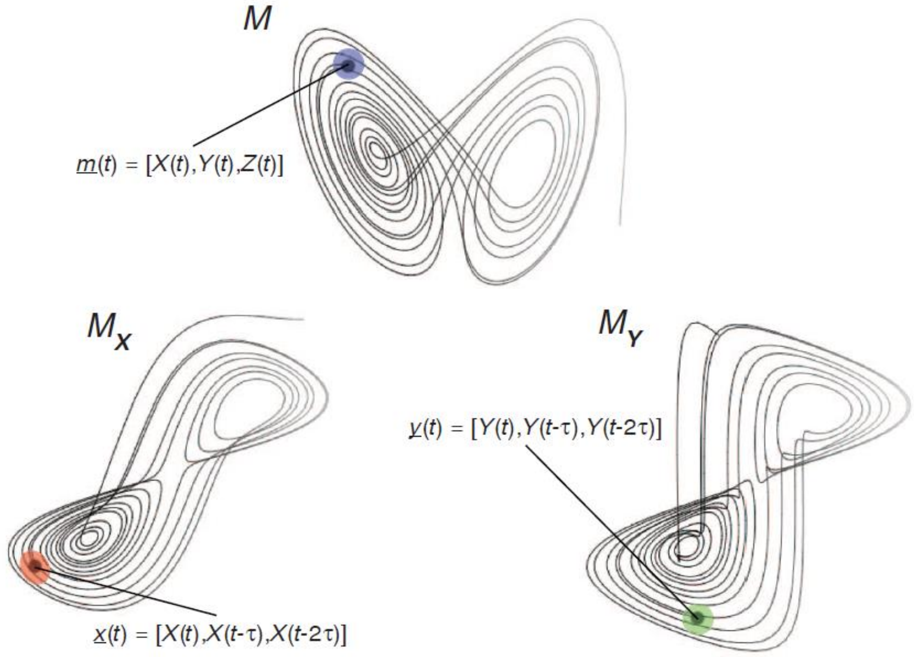
Conditional recurrence

Systematic application of those ideas: convergent cross-mapping (Sugihara et al., Science, 2012) – method for detecting causality among variables

$\Rightarrow$ Predictability of dynamical state space neighbors of one variable from dynamical neighbors of the other implies causal effect



Take home messages

Recurrence as a fundamental paradigm of dynamical systems

Recurrence plots to qualitatively distinguish different types of dynamics

Recurrence quantification analysis and recurrence network analysis – suite of measures of dynamical and geometric complexity to characterize nonlinear time series quantitatively

Bi- and multivariate extensions of recurrence plots, recurrence quantification analysis and recurrence networks to study interdependency between time series reflecting different subsystems or different variables (coupling identification and quantification, synchronization detection, direct vs. Indirect coupling, causality)

Cautionary note: not all variables are equally suited for recurrence analysis to provide interpretable information – depends (among others) on observability of associated variable

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pyunicorn

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25TH ANNIVERSARY

Unified functional network and nonlinear time series analysis for complex systems science: The pyunicorn package

Jonathan F. Donges, Jobst Heitzig, Boyan Beronov, Marc Wiedermann, Jakob Runge, Qing Yi Feng, Liubov Tupikina, Veronika Stolbova, Reik V. Donner, Norbert Marwan, Henk A. Dijkstra, and Jürgen Kurths

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