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Complexity and Turbulence in Space Plasmas

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An introduction to kinetic plasma theories

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Topics

- *Introduction*
- *Microscopic description: the Klimontovich equation*
- *The Liouville equation and the BBGKY hierarchy*
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- *The Mean Field Approximation*
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- *The Vlasov equation*
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Introduction

In treating several processes occurring in real magnetised space plasmas the *macroscopic description,* provided by the magnetohydrodynamic (MHD) (single fluid) approach could be not sufficient.

Indeed, MHD approach is a continuous media description (a field theory) which loose the particle nature of plasma media, and which is essentially based on a limited number of evolution equations for local macroscopic quantities (density, velocity, temperature,….).

These quantities are not representative of the full information contained in the full particle phase-space distribution function (DF) f_S(**r**, **v**; *t*), being these quantities essentially related to the first moments of the DF.

$$M^n f_s = \int \vec{v}^n f_s(\vec{r}, \vec{v}; t) d^3 v$$

e.g.,

$$M^0 f_s \equiv n_s(\vec{r}, t) = \int f_s(\vec{r}, \vec{v}; t) d^3 v \qquad M^1 f_s \equiv \vec{\Gamma}_s(\vec{r}, t) = \int \vec{v} f_s(\vec{r}, \vec{v}; t) d^3 v$$

Introduction

Thus, in some situations it is necessary to provide a more detailed description, which is is related to the evolution of the full particle DF and contains more information on the dynamics of plasma media.

This description is provided by *kinetic theories*, which can be considered as an intermediate description between the the formally exact *microscopic* description of plasma behaviour and the *macroscopic* MHD single fluid description.

Exact microscopic description (Klimontovich equation- Liouville equation)

⇓

Kinetic description (kinetic equations: Vlasov, Landau, Boltzmann, Lenard-Balescu)

⇓

Multi-fluid description (e.g. two-fluid equations)

⇓

Single fluid description (MHD approach)

Microscopic description: Klimontovich equation

Let's start our description of plasma evolution in terms of an ensemble of individual particles.

As a first step we can introduce a *density function, $N_s(\boldsymbol{r}, \boldsymbol{v}; t)$,* in the Boltzmann's single-particle *phase space*, ($\boldsymbol{r}$, $\boldsymbol{v}$), for each species *s*, which provides the exact *number of particles* with velocity $\boldsymbol{v}$ at the location $\boldsymbol{r}$;

$$N_s(\mathbf{r}, \mathbf{v}; t) = \sum_{i=1}^{N_0} \delta(\mathbf{r} - \mathbf{R}_i)\delta(\mathbf{v} - \mathbf{V}_i)$$

where ($\mathbf{R}_i$, $\mathbf{V}_i$) are the *Lagrangian* coordinates of the particles of the species *s* and ($\mathbf{r}$, $\mathbf{v}$) the *Eulerian* coordinates of the 6*d phase space.*

$$N(\mathbf{r}, \mathbf{v}; t) = \sum_s N_s(\mathbf{r}, \mathbf{v}; t)$$

We note how this quantity is *inherently singular.*

Microscopic description: Klimontovich equation

Now, each particle of the plasma evolves according to the *Newton-Lorentz force law*,

$$m_s \dot{\mathbf{V}}_i(t) = q_s \left\{ \mathbf{E}^m[\mathbf{R}_i(t), t] + \mathbf{V}_i(t) \times \mathbf{B}^m[\mathbf{R}_i(t), t] \right\}$$

where the superscript *m* on the fields indicates that these fields are the *microscopic fields* acting on single particle.

These *microscopic fields* result from the superposition of the external fields and the self-consistent fields generated by the other particles and acting on the considered *i^th^*-particle at the time *t*. Furthermore, they are strongly and rapidly varying fields and obey the Maxwell equations:

$$\nabla \cdot \mathbf{E}^m(\mathbf{r}, t) = \frac{\rho^m(\mathbf{r}, t)}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B}^m(\mathbf{r}, t) = 0$$

$$\nabla \times \mathbf{E}^m(\mathbf{r}, t) = -\frac{\partial B^m(\mathbf{r}, t)}{\partial t}$$

$$\nabla \times \mathbf{B}^m(\mathbf{r}, t) = \mu_0 \mathbf{J}^m(\mathbf{r}, t) + \frac{1}{c^2} \frac{\partial E^m(\mathbf{r}, t)}{\partial t}$$

$$\rho^m(\mathbf{r}, t) = \sum_s q_s \int d^3v N_s(\mathbf{r}, \mathbf{v}; t)$$

$$J^m(\mathbf{r}, t) = \sum_s q_s \int d^3v \mathbf{v} N_s(\mathbf{r}, \mathbf{v}; t)$$

Microscopic description: Klimontovich equation

Moving from the previous equations we can derive an equation for the time evolution of the plasma, as described by the DF, by time derivative of the DF itself:

$$\frac{\partial N_s}{\partial t} = -\sum_i \mathbf{V}_i \cdot \nabla \delta[\mathbf{r} - \mathbf{R}_i(t)]\delta[\mathbf{v} - \mathbf{V}_i(t)] - \sum_i \frac{q_s}{m_s}(\mathbf{E}^m + \mathbf{V}_i \times \mathbf{B}^m) \cdot \nabla_v \delta[\mathbf{r} - \mathbf{R}_i(t)]\delta[\mathbf{v} - \mathbf{V}_i(t)]$$

that, exchanging $\boldsymbol{V}_i \rightarrow \boldsymbol{v}$ (due to delta-function property), reduces

$$\frac{\partial N_s}{\partial t} = -\mathbf{v} \cdot \nabla N_s - \frac{q_s}{m_s}(\mathbf{E}^m + \mathbf{v} \times \mathbf{B}^m) \cdot \nabla_v N_s$$

or in terms of a *conservation law:*

$$\boxed{\frac{D}{Dt}N_s = \frac{\partial N_s}{\partial t} + \mathbf{v} \cdot \nabla N_s + \frac{q_s}{m_s}(\mathbf{E}^m + \mathbf{v} \times \mathbf{B}^m) \cdot \nabla_v N_s = 0}$$

This is the *Klimontovich equation*.

Microscopic description: Klimontovich equation

Klimontovich equation is *exact* and should be combined with the Maxwell equations for the *microscopic* electric, $\boldsymbol{E}^m$, and magnetic, $\boldsymbol{B}^m$, fields.

The equation expresses the *conservation* of the DF along a *particle path*:

$$\frac{D}{Dt} N_s = 0$$

which is equivalent to say that *plasma is incompressible.*

The *Klimontovich equations* are practically not solvable, being equivalent to solve all the particle motion. Furthermore, they contain all the information that may be too much to describe the plasma evolution at the investigated spatial and time scales.

A way to overcome this problem is to average the *DF* over an ensemble of replicas of the original system introducing the *phase-space averaged (reduced) distribution function f_s*

$$f_s(\mathbf{r}, \mathbf{v}; t) = \langle N_s(\mathbf{r}, \mathbf{v}; t) \rangle$$

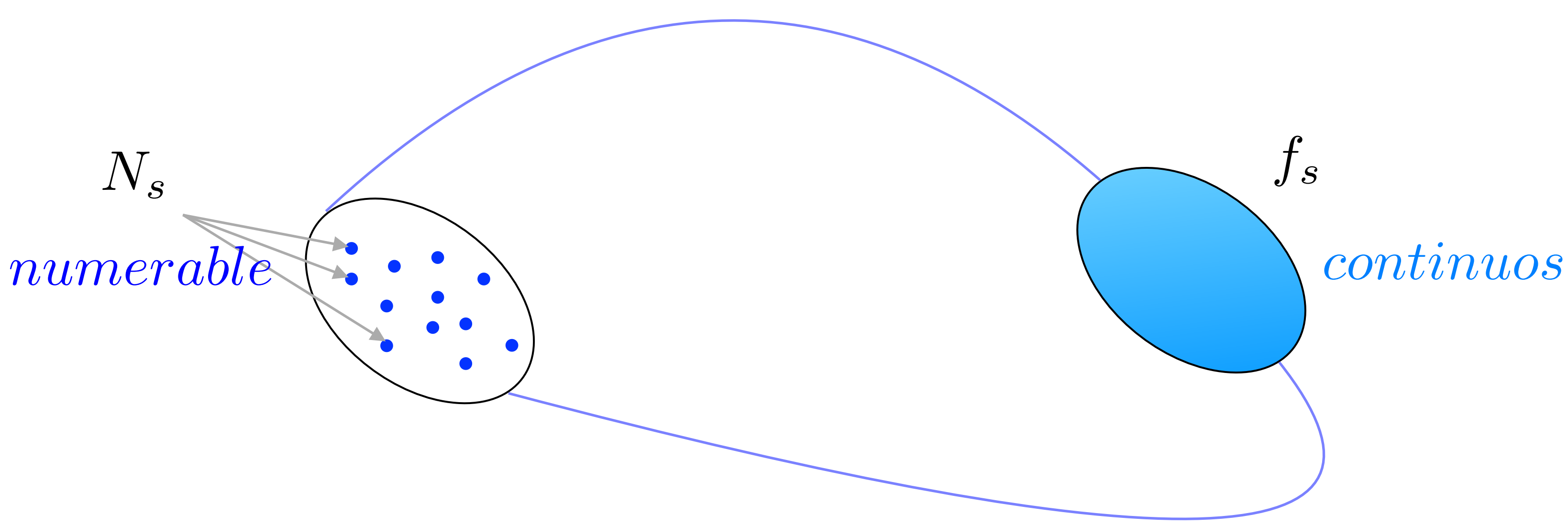
Microscopic description: Klimontovich equation

Thus, the *averaged (reduced) distribution function (RDF)* fs can also be interpreted as the number of particle in a certain interval of the phase space, i.e.

$$f_s(\mathbf{r},\mathbf{v};t) = \frac{\iint_{\Delta\Omega} N_s(\mathbf{r},\mathbf{v};t)d\Omega}{\Delta\Omega}$$

where $\Delta\Omega = \Delta x \Delta y \Delta z \Delta v_x \Delta v_y \Delta v_z$

This operation, which is equivalent to move from a singular function to a continuous function, has a non-trivial drawback [Ageno, 1995].



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Microscopic description: Vlasov equation

Let us now write exact functions and fields in terms of average ones and fluctuations, i.e.

$$N_s = f_s + \delta N_s$$
$$\mathbf{E}^m = \mathbf{E} + \delta \mathbf{E}^m$$
$$\mathbf{B}^m = \mathbf{B} + \delta \mathbf{B}^m$$

where

$$\mathbf{E} = \langle \mathbf{E}^m \rangle$$
$$\mathbf{B} = \langle \mathbf{B}^m \rangle$$

and

$$\langle \delta N_s \rangle = 0$$
$$\langle \delta \mathbf{E}^m \rangle = 0$$
$$\langle \delta \mathbf{B}^m \rangle = 0$$

We then can average the *Klimontovich equation* over the ensemble, getting

Plasma kinetic equation

$$\underbrace{\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s}_{collective\ effects\ large\ scale} = \underbrace{-\frac{q_s}{m_s}\langle (\delta \mathbf{E}^m + \mathbf{v} \times \delta \mathbf{B}^m) \cdot \nabla_v \delta N_s \rangle}_{collisional\ effects}$$

Microscopic description: Vlasov equation

$$\underbrace{\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s}_{\frac{Df_s}{Dt}} = \underbrace{-\frac{q_s}{m_s}\langle(\delta\mathbf{E}^m + \mathbf{v} \times \delta\mathbf{B}^m) \cdot \nabla_v \delta N_s\rangle}_{\left(\frac{\partial f_s}{\partial t}\right)_{coll}}$$

$$\frac{Df_s}{Dt} = \left(\frac{\partial f_s}{\partial t}\right)_{coll}$$

The relevance of the collisional effects can be estimated as follows:

$$\nu_c \sim \frac{nZ^2e^4}{2\pi\epsilon_0^2 m_e^2 v_0^3} \ln \Lambda$$

$$v_0 \sim \sqrt{\frac{k_B T_e}{m_e}} \qquad \frac{\nu_e}{\omega_{pe}} = \frac{2 \ln \Lambda}{3N_D} \underset{N_D \to \infty}{\longrightarrow} 0 \quad \Rightarrow \quad \left(\frac{\partial f_s}{\partial t}\right)_{coll} \longrightarrow 0$$

$$\Lambda = \frac{\lambda_D}{b_0} \sim N_D$$

Microscopic description: Vlasov equation

Under the previous limit the *kinetic equation* reduces to the well-known *Vlasov equation*

$$\frac{Df_s}{Dt} = 0 \longrightarrow \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s = 0$$

This equation has to be coupled with the *Maxwell equations* for the magnetic and electric field

$$\nabla \cdot \mathbf{E}(\mathbf{r};t) = \frac{1}{\epsilon_0} \rho(\mathbf{r};t)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r};t) = 0$$

$$\nabla \times \mathbf{E}(\mathbf{r};t) = -\frac{\partial \mathbf{B}(\mathbf{r};t)}{\partial t}$$

$$\nabla \times \mathbf{B}(\mathbf{r};t) = \mu_0 \mathbf{J}(\mathbf{r};t) + \frac{1}{c^2} \frac{\partial \mathbf{E}(\mathbf{r};t)}{\partial t}$$

$$\rho(\mathbf{r};t) = \sum_s q_s \int d^3v f_s(\mathbf{r},\mathbf{v};t)$$

$$\mathbf{J}(\mathbf{r};t) = \sum_s q_s \int d^3v \mathbf{v} f_s(\mathbf{r},\mathbf{v};t)$$

Vlasov-Maxwell equations for non-collisional plasmas

Microscopic description: Vlasov equation

Let us see some considerations and limits on *Vlasov equation* [Vlasov, 1938].

$$\frac{Df_s}{Dt} = 0 \longrightarrow \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_s = 0$$

Vlasov equation has the form of a *Liouville equation* for a set of ***non-interacting particles*** *moving under the action of an* ***external macroscopic field***.

The *effect of interaction* enters only in the definition of the *self-consistent average fields*, which are determined by the instantaneous value of the RDF, so that it is particularly important for describing long range interactions (e.g. Coulomb interactions in fully ionized plasmas).

The *Vlasov equation* implies a *non-trivial evolution* only for *spatially inhomogeneous systems* [Balescu, 2000].

A better definition of *Vlasov equation* would be that of a

Mean Field Equation

Liouville equation and BBGKY hierarchy

In the previous discussion we have derived the *Klimontovich equation* and the corresponding *kinetic equation* by adopting a 6d phase space description. This approach does not allow to correctly appreciate and describe the *right-hand collisional term* in terms of collisions and correlations.

A different method can be derived using an approach that moves from a description of the system evolution in the *full particle phase-space*: the *6Nd Gibbs phase-space (the Γ- phase space)*.

Let us consider a system of *N* interacting particles and indicate the set of dynamic *coordinates* ($\boldsymbol{q}_j$, $\boldsymbol{p}_j$) with x_j.

The system *Hamiltonian* is thus a function of all the particle dynamic coordinates,

$$H(x_1, x_2, ..., x_N) = \sum_{j=1}^{N} H^0(x_j) + \sum_{j<} \sum_{k=1}^{N} V(x_j, x_k),$$

where $V_{ij} = V(x_i, x_j)$ is the interaction potential.

Liouville equation and BBGKY hierarchy

In statistical mechanics the state of the system is determined by the *distribution function (DF) in the phase space* $F(x_1, ...x_N, t)$ which is defined over the 6Nd phase space.

It is useful to introduce also a set of *reduced distribution functions (RDFs)*, f_s, which refer to the distribution function for *s particles,* directly from the DF

For instance, in the case of the *single particle DF* we can write

$$f_1(x_1) = N \int dx_2 ... dx_N F(x_1, ..., x_N)$$

This point can be generalized to the case $s \leq N$

$$f_s(x_1, ..., x_s) = \frac{N!}{(N-s)!} \int dx_{s+1} ... dx_N F(x_1, ..., x_N).$$

RDFs are not independent, indeed if $r < s \leq N$, we get

$$f_r(x_1, ..., x_r) = \frac{(N-s)!}{(N-r)!} \int dx_{r+1} ... dx_s f_s(x_1, ..., x_s)$$

Liouville equation and BBGKY hierarchy

The validity of this representation in statistical mechanics stands in the *thermodynamic limit* (*T*-lim)

$$T - \lim; \; N \to \infty, \; V \to \infty, \; \frac{N}{V} = n = cte$$

which reflects on the RDFs in the fact that for any *s* and any configuration of (x_1, x_2, ..., x_s) the RDF f_s tends to a finite limit in the *T*-lim.

In the case of *spatially homogeneous systems* the RDFs are characterized by *translational invariance*, so that, for instance, we can write for the single-particle RDF and the two-particle one the following expressions:

$$f_s(q_1 + a, ..., q_s + a; p_1, ..., p_s) = f_s(q_1, ..., q_s; p_1, ..., p_s)$$

$$\Downarrow$$

moment distribution function

$$f_1(q, p) = n\varphi(p),$$

$$f_2(q_1, q_2; p_1, p_2) = f_2(q_1 - q_2; p_1, p_2)$$

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2nd Part

Liouville equation and BBGKY hierarchy

Let us now write the equation relative to the evolution of the RDFs.

The evolution equation of the *DF* is the well-known *Liouville equation:*

$$\partial_t F(q,p;t) = [H(q,p), F(q,p)]_P \equiv \hat{L} F(q,p;t),$$

where *H* is the *Hamiltonian operator,* $[\ldots]_P$ the *Poisson brackets* and $\hat{L}$ the *Liouville operator.*

Indicating with $\boldsymbol{v}_j = \boldsymbol{p}_j/m$ the velocity of the *j-th* particle, the Liouville operator can be written in the following form:

$$\hat{L} = \hat{L}^0 + \hat{L}' = \sum_{j=1}^{N} \hat{L}_j^0 + \sum_{j<} \sum_{n=1}^{N} \hat{L}'_{jn},$$

where

$$\hat{L}_j^0 = -\mathbf{v}_j \cdot \nabla_j, \ \hat{L}'_{jn} = (\nabla_j V_{jn}) \cdot \partial_{jn}$$

$$\nabla_j \equiv \frac{\partial}{\partial \mathbf{q}_j}, \ \partial_j \equiv \frac{\partial}{\partial \mathbf{p}_j}, \ \partial_{jn} \equiv \partial_j - \partial_n.$$

Liouville equation and BBGKY hierarchy

Using the same notation of the *Klimontovich equation* the *Liouville equation* takes the form:

$$\frac{\partial F}{\partial t} + \sum_{i=0}^{N} \underbrace{\mathbf{V}_i \cdot \nabla_r}_{-\hat{L}_i^0} F + \sum_{i=0}^{N} \underbrace{\dot{\mathbf{V}}_i \cdot \nabla_v}_{-\hat{L}'_{jn}} F = 0$$

where

$$\dot{\mathbf{V}}_i(t) = \frac{q_s}{m_s} \left[\mathbf{E}^m + \mathbf{v}_i \times \mathbf{B}^m\right]$$

This equation is of the form of a *convective time derivative*, expressing that the *density in the phase space* is *incompressible*

$$\frac{D}{Dt} F(x_1, x_2, ..., x_N; t) = 0$$

In the case of an external field the *Liouville operator* will contain an additional term that can be written as follows:

$$\hat{L}^F = \sum_{j=1}^{N} \hat{L}_j^F = \sum_{j=1}^{N} (\nabla_j V_j^F) \cdot \partial_j$$

Liouville equation and BBGKY hierarchy

We can now derive the evolution equations for the RDFs f_s

$$\partial_t f_s(x_1, ..., x_s) = \partial_t \frac{N!}{(N-s)!} \int dx_{s+1}..dx_N F(x_1, ..., x_N; t)$$

$$= \frac{N!}{(N-s)!} \int dx_{s+1}..dx_N \left\{ \sum_{j=1}^{N} \hat{L}_j^0 F + \sum_{j<} \sum_{n=1}^{N} \hat{L}'_{jn} F \right\}.$$

which by simple algebra [see *Balescu*, 2000] leads to the *BBGKY hierarchy*:

$$Bogoliubov - Born - Green - Kirkwood - Yvon$$

$$\partial_t f_0 = 0,$$

$$\partial_t f_1(x_1) = \hat{L}_1^0 f_1(x_1) + \int dx_2 \hat{L}'_{12} f_2(x_1, x_2),$$

$$\partial_t f_s(x_1, ..., x_s) = \sum_{j=1}^{s} \hat{L}_j^0 f_s(x_1, ..., x_s) + \sum_{j<} \sum_{n=1}^{s} \hat{L}'_{jn} f_s(x_1, ..., x_s) + \sum_{j=1}^{s} \int dx_{s+1} \hat{L}'_{j,s+1} f_{s+1}(x_1, ..., x_{s+1}).$$

Liouville equation and BBGKY hierarchy

By neglecting the trivial equation for the zeroth order function, the *BBGKY hierarchy* reduces to the following set of *N* equations for the *RDFs*

$$\partial_t f_1(x_1) = \hat{L}_1^0 f_1(x_1) + \int dx_2 \hat{L}'_{12} f_2(x_1, x_2),$$

$$\partial_t f_2(x_1, x_2) = [\hat{L}_1^0 + \hat{L}_2^0] f_2(x_1, x_2) + \sum_{j<} \sum_{n=1}^{2} \hat{L}'_{jn} f_2(x_1, x_2) + \sum_{j=1}^{2} \int dx_3 \hat{L}'_{j,3} f_3(x_1, x_2, x_3),$$

....

....

$$\partial_t f_s(x_1, ..., x_s) = \sum_{j=1}^{s} \hat{L}_j^0 f_s(x_1, ..., x_s) + \sum_{j<} \sum_{n=1}^{s} \hat{L}'_{jn} f_s(x_1, ..., x_s) + \sum_{j=1}^{s} \int dx_{s+1} \hat{L}'_{j,s+1} f_{s+1}(x_1, ..., x_{s+1}).$$

This set of equations has a *hierarchical structure*, so that the determination of the s-particle RDF requires the knowledge of the (s+1)-particle one.

The information contained in the *DF F* is now smeared on the set of RDFs fs. Anyway, in several cases the knowledge of the first two RDFs is sufficient.

Liouville equation and BBGKY hierarchy

The chain/hierarchy of BBGKY equations can also be written as follows:

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} \left(\mathbf{v}_i \cdot \nabla_{r_i} f_s + \sum_{j \neq 1}^{s} \mathbf{a}_{ij} \cdot \nabla_{v_i} f_s + \frac{N-s}{V} \int d^3 r_{s+1} d^3 v_{s+1} \mathbf{a}_{i(s+1)} \cdot \nabla_{v_i} f_{s+1} \right) = 0$$

In the case of single particle RDF we can write:

$$\frac{\partial}{\partial t} f_1 + \mathbf{v}_1 \cdot \nabla_{r_1} f_1 + \frac{N-1}{V} \int d^3 r_2 d^3 v_2 \mathbf{a}_{12} \cdot \nabla_{v_1} f_2 = 0$$

$$f_1 = f_1(\mathbf{r}_1, \mathbf{v}_1; t) \qquad f_2 = f_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2; t)$$

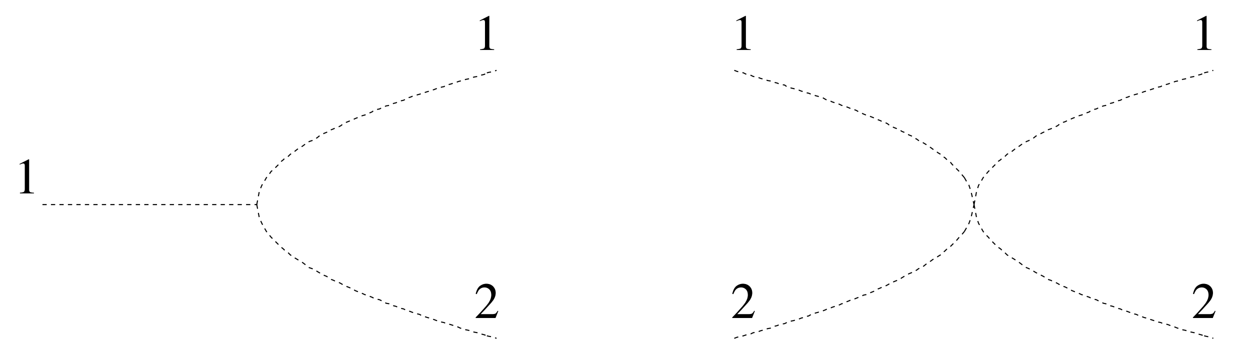
Liouville equation and BBGKY hierarchy

It is possible to introduce a *graphical representation* (a sort of *Feynman diagram*) for the interaction operators that appear in the RDFs equations.

An *s*-particle RDF can be thought as a set of *s* parallel horizontal lines, so that, while the free propagator (propagation operator) L^0 does not couple any of these lines, the interaction operator L'_{ij} can be represented as a vertex joining the *j* and *i* lines, being representative of the coupling (interaction) between these two particle lines.

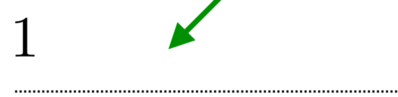
There are essentially two different type of vertex graphs:

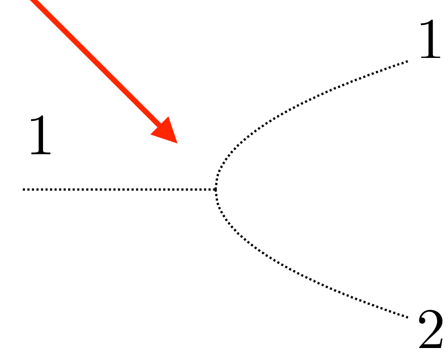
$Y - vertex$ $X - vertex$



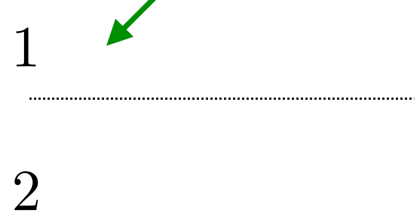
Liouville equation and BBGKY hierarchy

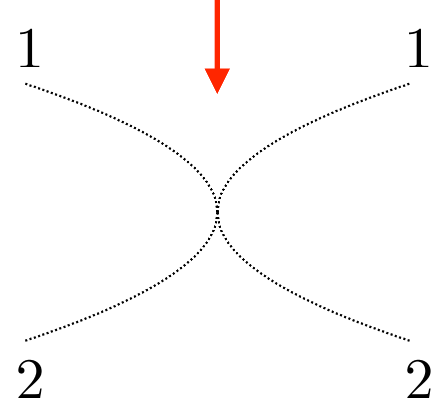
$$\partial_t f_1(x_1) = \hat{L}_1^0 f_1(x_1) + \int dx_2 \hat{L}'_{12} f_2(x_1, x_2),$$

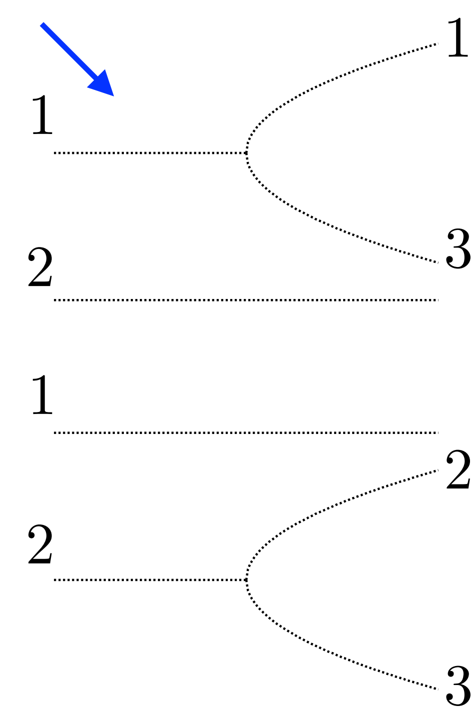




$$\partial_t f_2(x_1, x_2) = [\hat{L}_1^0 + \hat{L}_2^0] f_2(x_1, x_2) + \sum_{j<} \sum_{n=1}^{2} \hat{L}'_{jn} f_2(x_1, x_2) + \sum_{j=1}^{2} \int dx_3 \hat{L}'_{j,3} f_3(x_1, x_2, x_3),$$







Correlation functions and BBGKY hierarchy

The RDFs of 2 or more particles include *correlations* among particles.

Consider an *s*-particle RDF at a fixed time, then if there are *no-correlations* this could be written as the product of one-particle RDF, i.e.,

$$f_s^{unc}(x_1, ..., x_s; t) = \prod_{j=1}^{N} f_1(x_j; t),$$

The absence of *correlations* is analogous to *statistical independence*.

However, for real systems the presence of interactions generally implies the occurrence of a certain degree of correlation among the particles, which can extend up to the largest scales (i.e., at scales larger than the interaction range) as *cooperative effects.*

Correlations can be introduced by writing the s-particle RDFs as the sum of two terms:

$$f_s(x_1, ..., x_s; t) = f_s^{unc}(x_1, ..., x_s; t) + g_s'(x_1, ..., x_s) = \prod_{j=1}^{N} f_1(x_j; t) + g_s'(x_1, ..., x_s)$$

Correlation functions and BBGKY hierarchy

The quantity $g'(x_1,\dots, x_s;t)$ describes the effect of *correlations*, and can be written as the sum of terms, obtained by considering all the *partitions* of the *s* particles into *disjoint subsets* containing at least one particle.

For instance, this cluster representation in the case of 2 and 3 particles RDFs allows to write the RDFs in the following form,

$$f_2(x_1, x_2) = f_1(x_1)f_1(x_2) + g_2(x_1, x_2)$$

$$\begin{aligned} f_3(x_1, x_2, x_3) =& f_1(x_1)f_1(x_2)f_1(x_3) + f_1(x_1)g_2(x_2, x_3) \\ &+ f_1(x_2)g_2(x_3, x_1) + f_1(x_3)g_2(x_1, x_2) \\ &+ g_3(x_1, x_2, x_3). \end{aligned}$$

Before moving to find an evolution equation for the *correlations functions*, let us see a parallel between the above form of the 2 particle RDF and correlated statistical events.

$$P(x, y) = P(x)P(y) + \delta P(x, y)$$

$$\delta P(x, y) = P(x, y) - P(x)P(y)$$

$$\frac{\delta P(x, y)}{P(x)P(y)} = \frac{P(x, y)}{P(x)P(y)} - 1 = \frac{N^2}{N^2(N-1)} \sim \frac{1}{N}$$

Correlation functions and BBGKY hierarchy

To find the evolution equation for the $g'(x_1,\ldots,x_s;t)$ we can make use of the RDFs evolution equations and the BBGKY hierarchy. Computation is long and requires to use the graphs [see *Balescu*, 2000].

For example, for the two particle correlation function $g_2(x_1, x_2)$, we get,

$$\begin{aligned}\partial_t g_2(x_1,x_2) - [\hat{L}_1^0 + \hat{L}_2^0]g_2(x_1,x_2) = {} & \hat{L}'_{12}[f_1(x_1)f_1(x_2) + g_2(x_1,x_2)] \\ & + \int dx_3\{\hat{L}'_{13}f_1(x_1)g_2(x_2,x_3) + \hat{L}'_{23}f_1(x_2)g_2(x_1,x_3) \\ & + (\hat{L}'_{13} + \hat{L}'_{23})[f_1(x_3)g_2(x_1,x_2) + g_3(x_1,x_2,x_3)]\}\end{aligned}$$

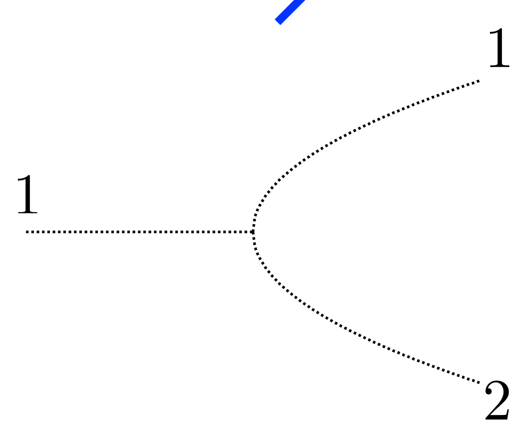
This equation has to couple to that of the single particle RDF to get the starting point for the development of the *kinetic equations and their approximations*

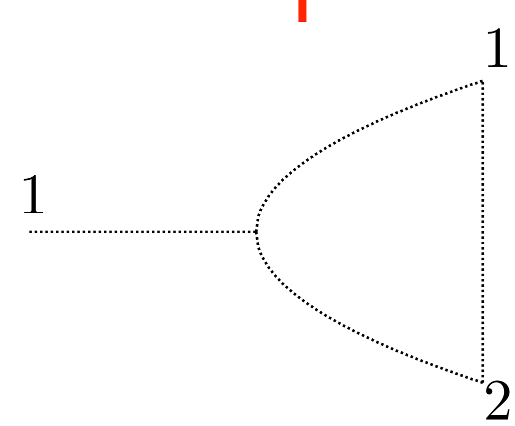
$$\partial_t f_1(x_1) - \hat{L}_1^0 f_1(x_1) = \int dx_2[\hat{L}'_{12}f_1(x_1)f_1(x_2) + \hat{L}'_{12}g_2(x_1,x_2)],$$

Correlation functions and BBGKY hierarchy

In terms of graphs the right hand term of the evolution equation of the single particle RDF can be represented as follows,

$$\partial_t f_1(x_1) - \hat{L}_1^0 f_1(x_1) = \int dx_2 [\hat{L}'_{12} f_1(x_1) f_1(x_2) + \hat{L}'_{12} g_2(x_1, x_2)],$$





Correlation functions and BBGKY hierarchy

$$\partial_t f_1(x_1) - \hat{L}_1^0 f_1(x_1) = \int dx_2 [\hat{L}'_{12} f_1(x_1) f_1(x_2) + \hat{L}'_{12} g_2(x_1, x_2)],$$

By inserting the expressions for the *Liouville operators* this equation can be written as

$$\frac{\partial}{\partial t} f_1 + \mathbf{v}_1 \cdot \nabla_{r_1} f_1 + \frac{N-1}{V} \int d^3 r_2 d^3 v_2 \mathbf{a}_{12} \cdot \nabla_{v_1} [f_1(x1) f_1(x2) + g_2(x_1, x_2)] = 0$$

and introducing the average acceleration

$$\mathbf{a}(\mathbf{q}_1; t) \equiv n_0 \int d^3 r_2 d^3 v_2 \mathbf{a}_{12} f_1(x_2; t)$$

we get

$$\partial_t f_1 + \mathbf{v}_1 \cdot \nabla_{r_1} f_1 + \mathbf{a} \cdot \nabla_{v_1} f_1 = -n_0 \int d^3 r_2 d^2 v_2 \mathbf{a}_{12} \cdot \nabla_{v_1} g_2$$

Mean Field Approximation

In several real situations the full hierarchy of evolution equations for the RDFs and the correlation functions can be significantly simplified.

Practically, correlation functions of order higher than a certain S and be neglected, although correlation functions of low order ($s \leq S$) can be represented in terms of the single particle RDF.

This operation will lead to describe the evolution of some systems by a *closed equation for the one particle RDF* $f_1(x_1; t)$, which is called

"*Kinetic Equation*"

To obtain the *kinetic equations* it is generally used a *perturbative approach,* starting from a *reference state* characterized by the absence of *any interaction.* This corresponds to do a perturbative expansion in the interaction potential which is supposed to be small, i.e.,

$$V(r) = \lambda v(r) \longrightarrow \lambda << 1, \ \left| \frac{v(r)}{v(r_0)} \right| = O(1), \ \forall r$$

$$weakly\ coupled\ system$$

Mean Field Approximation

This mean-field approach based on the hypothesis of a *weak-coupling* is particularly valid for treating systems like *space plasmas* (or plasmas in general), which are characterized by small long range interactions (at scales larger than the Debye length).

This approximation reflects on the *Liouville operators* and the *RDFs* and the *correlation functions,*

$$\hat{L}_j^0 = O(\lambda^0), \quad \hat{L}'_{jm} = O(\lambda).$$

$$f(x_1;t) = O(\lambda^0), \quad g_2(x_1, x_2; t) = O(\lambda), \quad g_3(x_1, x_2, x_3; t) = O(\lambda^2).$$

Another relevant question deals with the existence of a *correlation length* l_C for the *correlation functions* g_s which is connected with the fact that in several cases the interaction potential has a limited range of action:

$$V(\mathbf{q}_1, \mathbf{q}_2) = V(r_{12}) \qquad V(r) \approx 0 \;\leftrightarrow\; r \gg l_0$$

$yardstick\ length$

$$l_C = \max(l_0, l_{C2}, l_{C3}, ...)$$

Free particle dynamics

Let us consider a system of *non-interacting particles,*

$$\lambda = 0, \quad \hat{L}'_{jn} = 0$$

This is an *ideal system*, which is the starting point for the discussion and modeling of non-equilibrium description.

The starting equations reduces to the following form:

$$\partial_t f(x_1;t) = L_1^0 f(x_1;t),$$
$$\partial_t g_2(x_1,x_2;t) = (L_1^0 + L_2^0) g_2(x_1,x_2;t)$$

....

Thus, in a *free system* the equations for *RDFs* and *correlations functions* are not *coupled*, so that there is no way to *generate or destroy correlations.*

We can now introduce an *operator for free propagation* $U^0{}_1(t)$

$$U_1^0(t) = \exp(L_1^0 t) = \exp(-\mathbf{v}_1 \cdot \nabla_1 t)$$

Free particle dynamics

Thus,

$$f(\mathbf{q}_1, \mathbf{v}_1; t) = U_1^0(t) f(\mathbf{q}_1, \mathbf{v}_1; 0),$$

i.e.,

$$f(\mathbf{q}_1, \mathbf{v}_1; t) = f(\mathbf{q}_1 - \mathbf{v}_1 t, \mathbf{v}_1; 0).$$

being

$$\exp\left(a \frac{d}{dx}\right) f(x) = f(x + a)$$

Here (and in what follows), we adopt a representation in terms of velocity, instead of moment:

$$f(\mathbf{q}, \mathbf{p}; t) = m^3 f(\mathbf{q}, \mathbf{v}; t)$$

$$\int d\mathbf{q} d\mathbf{p} f(\mathbf{q}, \mathbf{p}; t) = \int d\mathbf{q} d\mathbf{v} f(\mathbf{q}, \mathbf{v}; t) = N.$$

$$x_i = (\mathbf{q}_i, \mathbf{v}_i), \ \partial_j = \partial / \partial \mathbf{v}_j, \ \partial_{jn} = \partial_j - \partial_n$$

The Vlasov Equation from BBGKY approach

The *Vlasov equation* can be obtained from the expansion of the *evolution operator* up to the order λ

Thus, writing the evolution equation of the first RDF f_1

$$\partial_t f(x_1) - \hat{L}_1^0 f(x_1) = \int dx_2 [\underbrace{\hat{L}'_{12} f(x_1) f(x_2)}_{\lambda\ (\sim \hat{L}' f f)} + \underbrace{\hat{L}'_{12} g_2(x_1, x_2)}_{\lambda^2\ (\sim \hat{L}' g_2)}],$$

we obtain at the order λ

$$\mathit{Vlasov\ Equation}$$

$$\boxed{\partial_t f(x_1;t) = \hat{L}_1^0 f(x_1;t) + \int dx_2 \hat{L}'_{12} f(x_1;t) f(x_2;t)}$$

the *Vlasov equation* which describes the evolution of the single-particle RDF in absence of correlations.

The Vlasov Equation from BBGKY approach

Let us now write the in a more clear way the second term in the right-hand:

$$\partial_t f(x_1;t) = \hat{L}_1^0 f(x_1;t) + \int dx_2 \hat{L}'_{12} f(x_1;t) f(x_2;t)$$

i.e.,

$$\nu\{f,f\} = m^{-1} \int dx_2 (\nabla_1 V_{12}) \cdot \partial_{12} f(x_1;t) f(x_2;t)$$

$$= m^{-1} \left\{ \nabla_1 \int d\mathbf{q}_2 d\mathbf{v}_2 V_{12}(\mathbf{q}_1 - \mathbf{q}_2) f(\mathbf{q}_2, \mathbf{v}_2; t) \right\} \cdot \partial_1 f(\mathbf{q}_1, \mathbf{v}_1; t)$$

where

$$\langle V(\mathbf{q}_1;t) \rangle = \int d\mathbf{q}_2 d\mathbf{v}_2 V_{12}(\mathbf{q}_1 - \mathbf{q}_2) f(\mathbf{q}_2, \mathbf{v}_2; t)$$

Thus, the quantity in the brackets is the *average force*, so that the *Vlasov equation* takes the form

$$(\partial_t + \mathbf{v} \cdot \nabla) f(\mathbf{q}, \mathbf{v}; t) = m^{-1} [\nabla \langle V(\mathbf{q};t) \rangle] \cdot \partial f(\mathbf{q}, \mathbf{v}; t)$$

The Vlasov Equation from BBGKY approach

Vlasov equation has the form of the *Liouville equation* in the case of *non-interacting particles (no-correlations)* moving under the action of an *external field*.

The external field has to be computed *synchronously,* so that the equation is *inherently nonlinear*.

The *Vlasov equation* provides a good description for the evolution of *spatially inhomogeneous systems* in which the *Vlasov mean field* $\langle V(\mathbf{q};t)\rangle$ has an *action range* l_c larger than the typical *density gradient scale* l_H.

That is why *Vlasov equation* is the starting point for the description of *non-collisional plasmas*, where due to the long range action of potential the force acting on the particles is the result of a collective phenomenon ($N_D \to \infty$).

In presence of on electromagnetic field the *Vlasov equation* can be written

$$(\partial_t + \mathbf{v}\cdot\nabla) f(\mathbf{q},\mathbf{v};t) + \frac{e}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\partial f(\mathbf{q},\mathbf{v};t) = 0$$

$$\mathbf{E} = \mathbf{E}_0 + \langle\mathbf{E}\rangle = -\nabla(\Phi_0 + \langle\Phi\rangle) = 0$$

More…….

As we have widely discussed, *Vlasov average potential* (which appears in the Vlasov eq.) properly describes some of the features of the *long range nature of the Coulomb potential*, which is able to describe the *collective behavior* of plasma systems.

However, *Vlasov equation* is not sufficient to describe the *irreversible evolution towards equilibrium* of plasma systems, being the *correlation term completely removed*.

Indeed, *Vlasov equation* does not satisfy the *Boltzmann's H-Theorem* for entropy production:

$$\frac{ds(t)}{dt} \geq 0$$

where

$$s(t) = -\kappa_B \int d^3v \ln[n\varphi(\mathbf{v};t)]\varphi(\mathbf{v};t) + c$$

How can we describe collision and the evolution in a plasma which has long range interaction ?

More…….

Now the derivation of a *kinetic equation for plasmas including correlations and collisions* requires some that assumptions on the interaction potential have to be satisfied.

In particular, it is required that interactions and correlations *have to be of a finite range.*

This point is in principle not satisfied by Coulomb potential which is of infinite range, and introduces *infrared and ultraviolet divergences.*

The *ultraviolet (short-range) divergence* introduced by the pure Coulomb potential can be overcome by assuming that at short distances a non-Coulombian repulsive interaction (e.g, of quantomechanical origin) can act, while the *infrared (long-range) divergence* can be removed by the role that *polarization effects* play in plasmas (Debye screening).

Thus, the effect of collision integral in plasma can be treated by replacing the pure Coulomb potential, by the Debye-Hückel potential

$$V^D(r) = e^2 \frac{\exp(-k_D r)}{r} \qquad \tilde{V}^D(k) = \frac{e^2}{2\pi} \frac{1}{k^2 + k_D^2}$$

More…….

By treating the *Coulomb interaction* as a weak interaction (i.e., e^2 is a small parameter, that can be equivalent to a *λth-order*), it is possible to derive a modified version of the *Vlasov equation* which includes also *collision and correlation effects.*

The starting point is to consider the evolution equation for the single-particle RDF and for the correlation function:

$$(\partial_t - L_1^0) f_1^\alpha(x_1;t) = \sum_\beta \int dx_2 \left[L_{12}^{'\alpha\beta} f_1^\alpha(x_1;t) f_1^\beta(x_2;t) + L_{12}^{'\alpha\beta} g_2^{\alpha\beta}(x_1,x_2;t) \right]$$

$$\begin{aligned}(\partial_t - L_1^0 - L_2^0) g_2^{\alpha\beta}(x_1,x_2;t) =& L_{12}^{'\alpha\beta} f_1^\alpha(x_1;t) f_1^\beta(x_2;t) \\ &+ \sum_\gamma \int dx_3 \Big\{ L_{13}^{'\alpha\beta} \left[f_1^\alpha(x_1;t) g_2^{\beta\gamma}(x_2,x_3;t) + f_1^\gamma(x_3;t) g_2^{\alpha\beta}(x_1,x_2;t) \right] \\ &+ L_{23}^{'\beta\gamma} \left[f_1^\beta(x_2;t) g_2^{\alpha\gamma}(x_1,x_3;t) + f_1^\gamma(x_3;t) g_2^{\alpha\beta}(x_1,x_2;t) \right] \Big\}\end{aligned}$$

More…….

Solution of these two coupled equations considering the previous potential and assuming that the plasma system is homogeneous,

$$f_1^{\alpha}(x_1;t) = n\varphi^{\alpha}(\mathbf{v}_1;t) \equiv n\varphi^{\alpha}(1)$$

$$g_2^{\alpha\beta}(x_1,x_2;t) = g_2^{\alpha\beta}(\mathbf{q}_1 - \mathbf{q}_2, \mathbf{v}_1, \mathbf{v}_2; t)$$

leads to the *Balescu-Lenard kinetic equation for plasmas*

$$\begin{aligned}\partial_t\varphi^{\alpha}(\mathbf{v}_1;t) =& 8\pi^4 n e_\alpha^2 \sum_\beta e_\beta^2 \int d^3v_2 \int d^3k\mathbf{k}\cdot\partial_1^{\alpha} \\ & \times \frac{\tilde{V}^2(k)}{\mid \epsilon(v_1) \mid^2}\delta(\mathbf{k}\cdot(\mathbf{v}_1-\mathbf{v}_2))\mathbf{k}\cdot\partial_{12}^{\alpha\beta}\varphi^{\alpha}(\mathbf{v}_1;t)\varphi^{\alpha}(\mathbf{v}_2;t)\end{aligned}$$

$dielectric\ function$

$$v_1 = \frac{\mathbf{k}\cdot\mathbf{v}}{k}$$

More…….

The form of the *Balescu-Lenard equation* reminds the form of *Landau kinetic equation* (weak interaction) where the *Coulomb interaction potential* has been replaced by an *effective potential (the screened potential)*.

Plasma is, thus, *weakly coupled system* where *collisions occur via an effective screened potential*, derived exactly by dynamical laws.

The *collective effects* are contained in the *screened potential*, which is not, here, considered to be static (as in *Vlasov equation*) but conversely is a *functional* of the *instantaneous plasma state.*

Here, *collisions involve all the plasma medium.*

In spite of the simple final form of *Balescu-Lenard equation* the nonlinearity contained into the equation is extremely difficult to manage.

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