Granularity Meets Smoothness

Stochastic Approaches to (Space) Plasmas

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Granularity Meets Smoothness: Stochastic Approaches to (Space) Plasmas*

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Abstract

Plasmas are systems of many, electrically charged particles interacting with each other through electromagnetic fields. As the number of point particles composing the plasma is of the order of (great) powers of ten, it comes natural to treat such systems not using directly their (too many!) *mechanical* degrees of freedom pertaining to particles. From the description of the position and momentum of each particle of the plasma, one passes to Gibbs or Boltzmann distributions, renouncing to describe the single particle details in favour of a *statistical picture*. The fundamental ingredients of statistical descriptions are still attributions of [illegible] particles or group of particles, but the individuality of each particle is disregarded in favour of that of "representative" constituents [illegible] nature.

Going ahead in this "zooming out" process, one [illegible] to [illegible] theories: the most widely used description of space plasmas (and also of many other plasma systems, e.g. the fusion plasmas in tokamaks) is Magneto-Hydro-Dynamics (MHD), that is essentially a classical field theory [illegible] each point in the space $\vec{x}$ two fields are defined, depending on time t, namely the magnetic induction field $\vec{B}$ and the [illegible] bulk velocity $\vec{V}$. MHD is not a statistical theory any more, it [illegible] one, at the price of regarding the plasma as a smooth, continuous [illegible] of matter, where the individuality of particles is definitely lost and nothing is retained of the granular nature of matter.

Needless to mention, MHD applicability is incredibly wide, and the results due to its use are paramount. Yet, there exist some physical situations in which representing the plasma as a "smooth continuum" may fail.

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STUDENT ADVISORY: EXPLICIT CALCULATIONS!

Primarily, in the presence of *turbulence* the macroscopic quantities describing the plasma at MHD scales do show *irregularity*, i.e. closely resemble the time evolution or space distribution of *non-differentiable fields*.

The physical point explaining those limitations of the theory, is that the MHD continuum is what remains of a crowd of particles, i.e. of an intrinsically different system, able to show a very different intrinsic behaviour, in which the *granular nature of matter* is not negligible.

These lecture present two attempts of enabling the MHD continuum to mimic some of the behaviours of the crow of particles, recognizing the mathematical [illegible] of suitable MHD [illegible], as the latter are replaced by noise [illegible]. As [illegible] noise terms are [illegible] them, MHD equations are [illegible] *stochastic field* [illegible] namely Langevin equations: [illegible] dynamical variables [illegible] become stochastic variables [illegible] and their [illegible] treated via the *functional formalism* of classical statistical [illegible]. This is the core subject of the presentation here [illegible] such framework, the MHD fields undergo a *multi-history probabilistic evolution*; one obtains a proper *Stochastic Field Theory* (SFT).

[illegible] examples show two theories, usually describing "smooth" plasmas, may be transformed into SFTs.

The first one is ordinary *resistive MHD*, in which the terms describing the density, the pressure and the conductivity of the plasma are irregular functions of space and time when turbulence occurs, and get replaced by noise terms.

The other one is *tetrad incompressible MHD*, taking into account the material nature of plasma through Lagrangian variables and enabling multi-scale interactions to be explicitly included. The probabilistic equations of motion of quantities describing the evolution of a turbulent plasma are constructed for tetrad MHD after recognizing suitable noise terms.

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Role and origin of noise.

• **Noise and multi-history evolution: statistical dynamics via functional formalism.**

PLAN OF THE LECTURES

LECTURE 2/2

• **Application #1**: stochastic field theory formulation of **resistive MHD**.

• **Application #2**: stochastic **tetrad dynamics for turbulent MHD** plasmas.

• Invitation to get curious: **fractal magnetic reconnection** and metriplectic **kinetic equation**

PLAN OF THE LECTURES

LECTURE 1/2

• **Motivation: turbulent variability in (almost) whatever concerns near-Earth plasma.**

Role and origin of noise;

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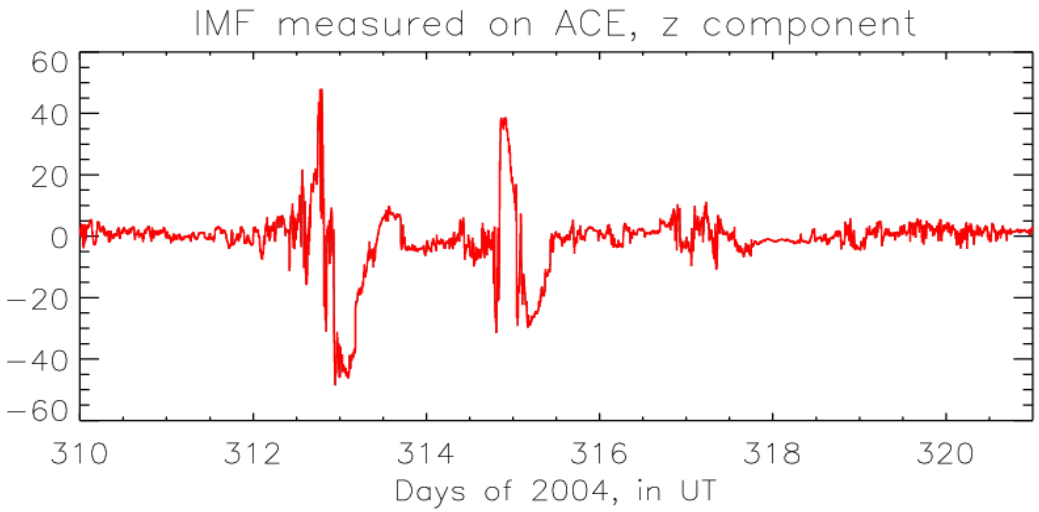
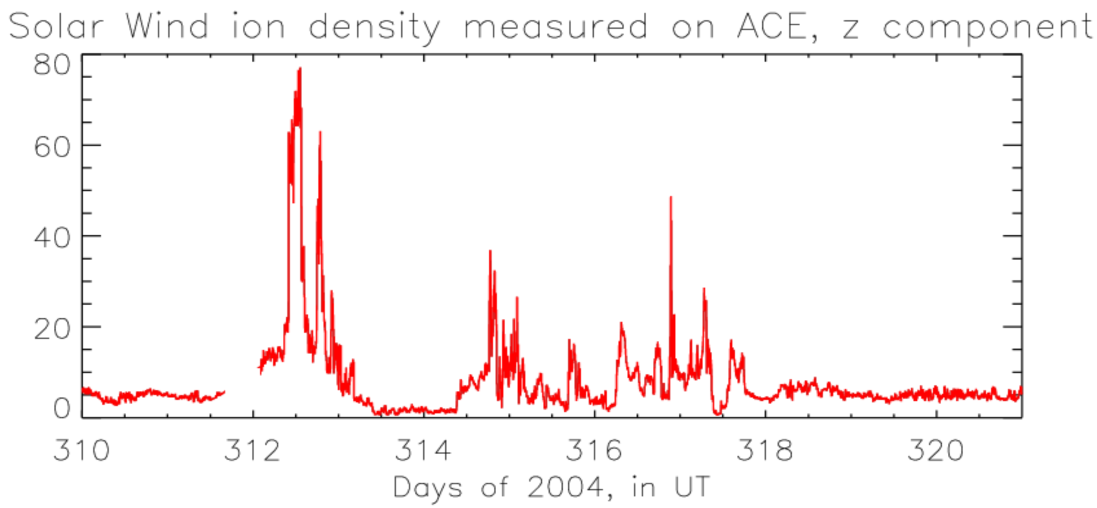
Space plasma is **a system of particles** interacting through electromagnetic forces. MHD represents it via **smooth and deterministic fields**. This representation **conceals the granularity of plasma**.

Still, granularity emerges whenever strong **gradients** or **impulsive phenomena** take place: hence, classical fields look **erratic and noisy**.

ENCODE GRANULARITY EFFECTS IN CONTINUA VIA NOISE TERMS





Materassi, Ciraolo, Consolini, Smith, ASR 47 (5), 2010

2 sources of noise in near-Earth plasma

http://geomag.org/info/magnetosphere.html

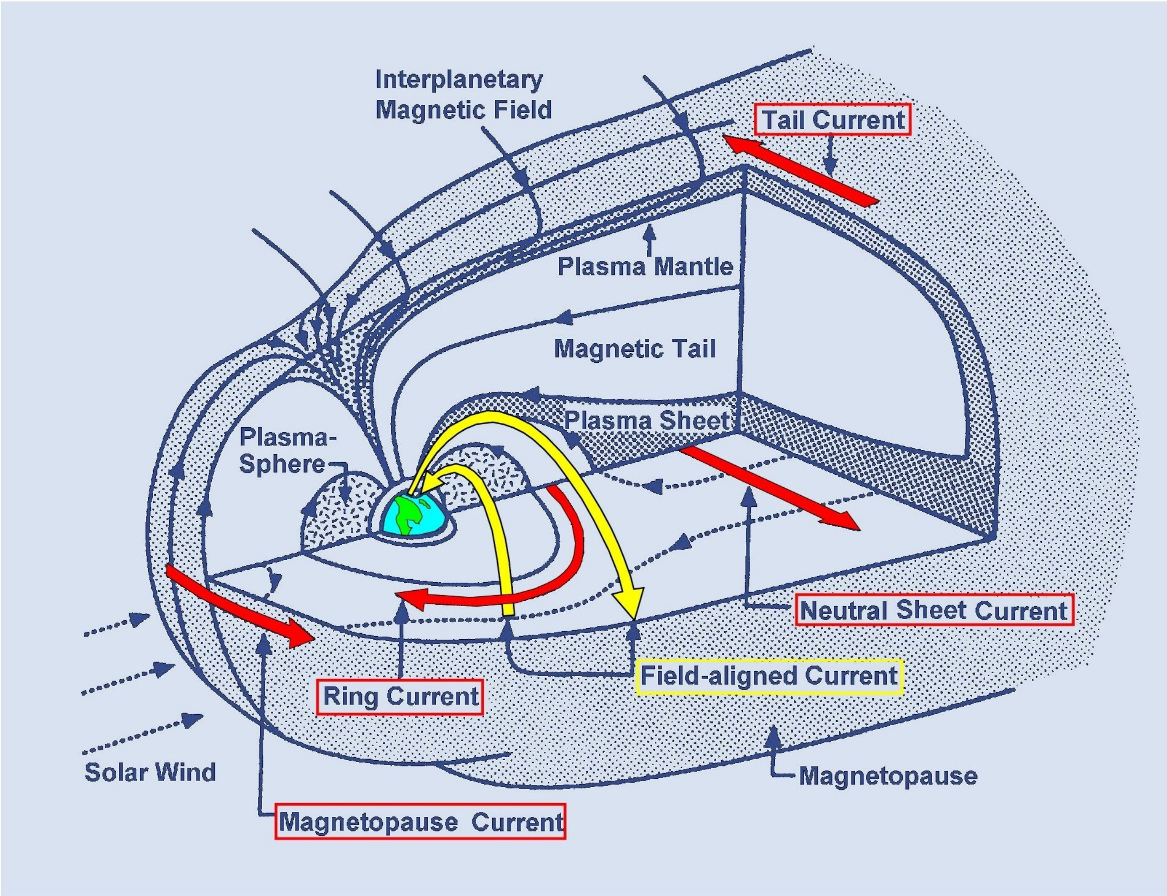


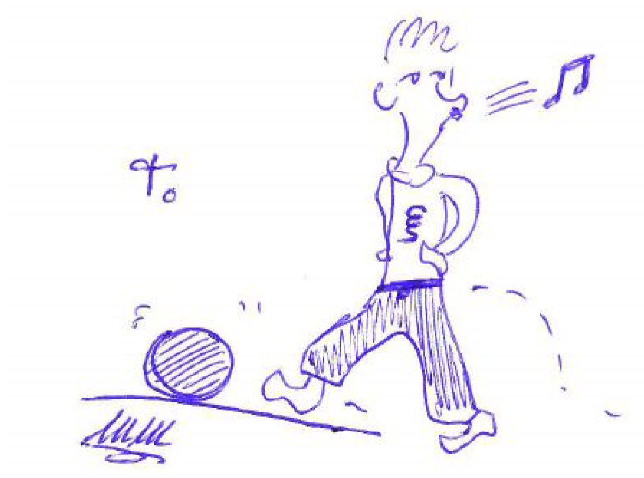
Fig. 1: A sketch of the magnetosphere (modified from Kivelson and Russel (1995)

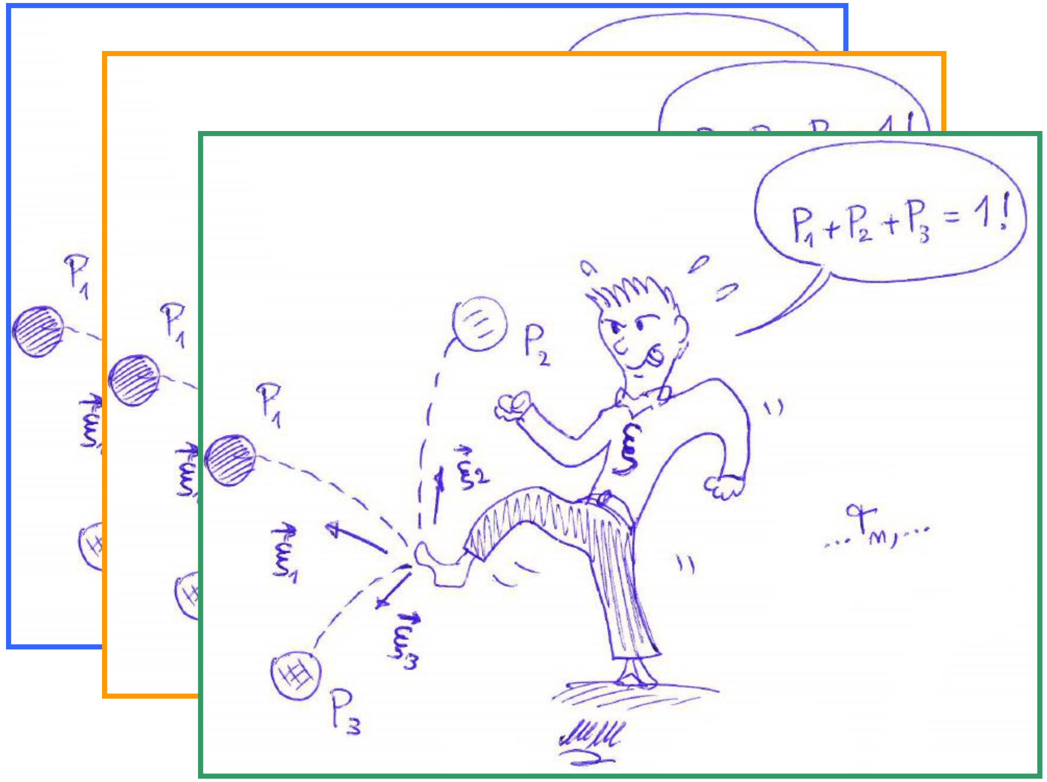
1)- The Solar Wind and Sun forcing on the system may be highly erratic: external noise

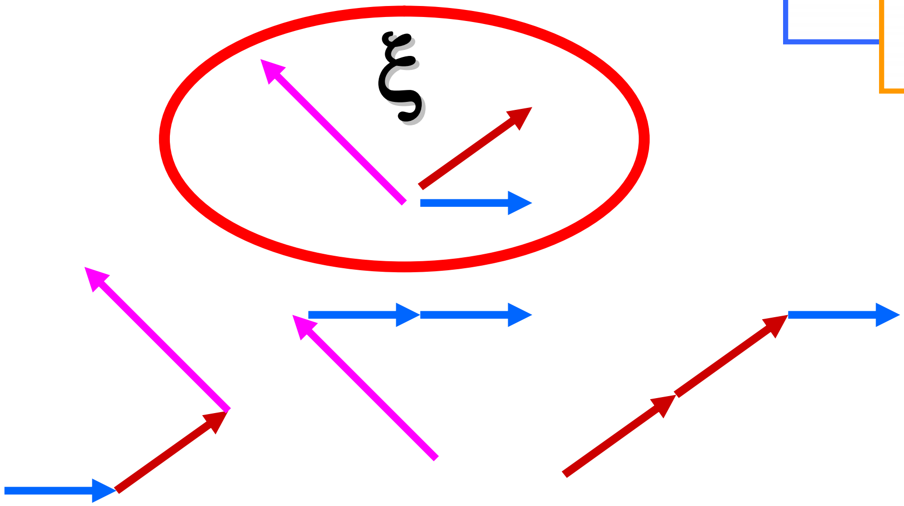
2)- Granularity of matter, bringing to the classical field scales particle phenomena: internal noise.

This is the noise more closely related to turbulence, and this is what we will deal with

The ***erratic ball***: random kicks ξ act on the **dynamical variables** (x,p), **handing them** their "stochasticity"







MANY DIFFERENT HISTORIES ARE (differently) PROBABLE!

$$\boxed{\mathcal{Q}\left(\vec{\xi},t\right) \mapsto \mathcal{P}\left(\vec{x},\vec{p},t\right)}$$

PLAN OF THE LECTURES

LECTURE 1/2

• Motivation: turbulent variability in (almost) whatever concerns near-Earth plasma.

Role and origin of noise.

• Noise and multi-history evolution: statistical dynamics via functional formalism.

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The functional formalism of classical statistical dynamics

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Abstract. A simple and general derivation of the functional formalism of classical statistical dynamics of Martin, Siggia and Rose is presented without the necessity of introducing non-commuting operators into the discussion. This is achieved by making use of functional integral representations of the correlation and response functions of the system. Some approximation procedures based on the functional integral representation are briefly discussed.

General framework for a **dynamical system**:

$$\psi : \mathbb{I} \mapsto \mathbb{V},\ \mathbb{I} = [t_0, t]$$

ψ is the state of the system at hand, V is its phase space

Example 1: the point particle (finite dimensional system)

$$\psi = (\vec{x}, \vec{p})\ ,\ \mathrm{T}_{(\vec{x},\vec{p})}(\mathbb{V}) = \mathbb{R}^6$$

Example 2: MHD in local variables (infinite dimensional system, field theory)

$$\psi(\vec{x}, \tau) = \left(\vec{B}(\vec{x}, \tau), \vec{V}(\vec{x}, \tau)\right),\ \mathbb{V} = \mathcal{C}\left(\mathbb{R}^3 \times \mathbb{I}, \mathbb{R}^6\right)$$

Noise stirred **dynamical systems:**

$$\dot{\psi}^I = \Lambda^I(\psi) + f^I + g_J \Gamma^{JI}(\psi)$$

Deterministic part: Λ is an expression in ψ, that's it...

Additive noise: at each time, f is the output of a stochastic process of given probability law

Multiplicative noise: at each time, f and g are the output of not necessarily uncorrelated stochastic processes. Γ is as deterministic as Λ

Time-local ODEs and PDEs will be dealt with (in most cases, also space-local)

$$Q_{f,g}\left(f^I(\tau), g_J(\tau)\right)$$

Finite dimensional stochastic dynamical systems (SDSs):

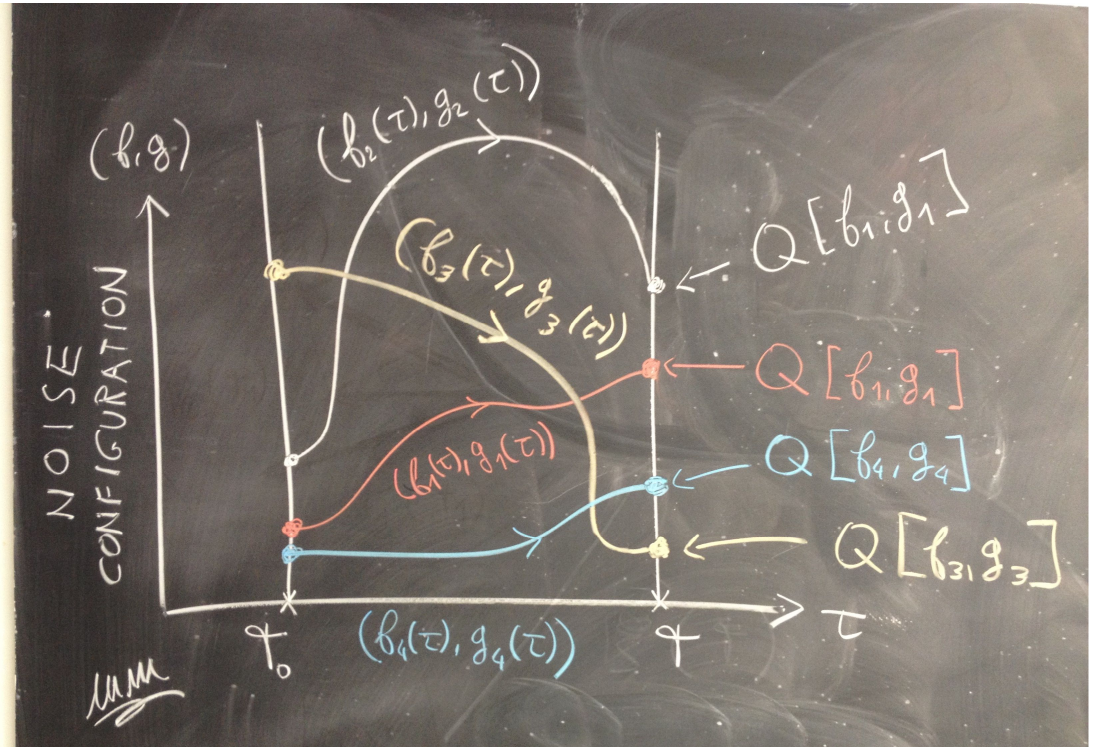
$$\dot{\psi}^I = \Lambda^I(\psi) + f^I + g_J \Gamma^{JI}(\psi)$$

Infinite dimensional SDSs (stochastic field theory, SFT):

$$\partial_\tau \psi^I(\vec{x},\tau) = \Lambda^I[\psi;\vec{x},\tau) + f^I(\vec{x},\tau) +$$

$$+\int g_J(\vec{y},\tau)\,\Gamma^{JI}[\psi;\tau,\vec{y},\vec{x})\,d^3y$$

Noises will evolve probabilistically in the time interval $[t_0, t]$. Each noise “history” can be assigned a probability to take place $Q[f,g]$.



Defining a probability measure in the sample space of possible evolutions.

In $Q[f,g]$ square brackets are used to underline the quantity depends on the value (f,g) at every time τ (infinite variables).

Probability over a densely-infinite-dimensional space.

Ensemble calculations are possible:

$$\langle F \rangle_{f,g} \stackrel{\text{def}}{=} \int_{\Phi} [df\, dg]\, \mathcal{Q}\,[f, g; t_0, t)\, F\,[f, g]$$

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Trying to give a sense to a functional integral (finite dimensional SDS) as the continuous limit of a multiple integral:

$$\langle F\left[f,g\right]\rangle_{f,g} = \lim_{N\to+\infty}\lim_{\Delta t\to 0}\left(\prod_{i=1}^{N}\int_{\mathbb{R}^n} df\left(\tau_i\right)\right)\left(\prod_{j=1}^{N}\int_{\mathbb{R}^n} dg\left(\tau_j\right)\right)$$

$$Q_N\left(\left(f\left(\tau_1\right),g\left(\tau_1\right)\right),\ldots,\left(f\left(\tau_N\right),g\left(\tau_N\right)\right)\right)F\left(\left(f\left(\tau_1\right),g\left(\tau_1\right)\right),\ldots,\left(f\left(\tau_N\right),g\left(\tau_N\right)\right)\right)$$

Initial condition (Cauchy) Problem for a SDS:

$$\begin{cases}\dot{\psi} = \Lambda\left(\psi\right) + f + g\cdot\Gamma\left(\psi\right), \\ \psi\left(t_0\right) = \psi_{\mathrm{i}}\end{cases}$$

Each ($f(\tau)$,$g(\tau)$) is a different collection of time-dependent coefficients in the ODEs: each "noise history" is a different ψ history (remember the "erratic ball").

Instead of calcuating the final configuration once the initial one is given at t_0, *thou shalt compute* the probability that the system has a certain configuration at a certain later time $t > t_0$.

The real object of desire of a SDS theory is the probability functional *A*[ψ], known *Q*[*f*,*g*]. Going from *Q* to *A* means going from the knowledge of noise statistics to that of system statistics.

This is done in two steps.

1. Equating the system ensemble statistics to noise statistics (because chance comes from noise):

$$\langle F \rangle_{\psi(f,g)} = \langle F \rangle_{f,g} \ \forall \ F[\psi]$$

2. Taking into account that what relates noises to system variables is dynamics:

$$\int_{\mathbf{\Omega}} [d\psi]\, \mathcal{A}[\psi; t_0, t)\, F[\psi] = \int_{\mathbf{\Phi}} [df\, dg]\, \mathcal{Q}[f, g; t_0, t)\, F[\psi(f, g)]$$

$$[d\psi]\, \mathcal{A}[\psi] \quad \overset{\dot{\psi}=\Lambda+f+g\cdot\Gamma}{\leftrightarrows} \quad [df\, dg]\, \mathcal{Q}[f, g]$$

SFT (key slide!): $\mathcal{A}[\psi]$ is expressed via an intermediate step kernel $A[\psi,\chi]$, depending on ancillary variables χ (stochastic momenta) and characteristic functional C of noises.

$$\partial_\tau \psi^I(\vec{x},\tau) = \Lambda^I[\psi;\vec{x},\tau) + f^I(\vec{x},\tau) + \int g_J(\vec{y},\tau)\,\Gamma^{JI}[\psi;\tau,\vec{y},\vec{x})\,d^3y,$$

$$\mathcal{A}[\psi;t_0,t) = \int [d\chi]\,A[\chi,\psi;t_0,t)\,,$$

$$A[\chi,\psi;t_0,t) = N_0(t_0,t)\,C[\chi,\Gamma;t_0,t)\,e^{-iS_0[\psi,\chi;t_0,t)},$$

$$C[\chi,\psi;t_0,t) = \int [df\,dg]\,\mathcal{Q}[f,g;t_0,t)\,e^{-iS_{f,g}[f,g,\psi,\chi;t_0,t)},$$

$$S_{f,g} = -\int_{t_0}^{t} d\tau \int d^3x\left[f^I(\vec{x},\tau)\,\chi_I(\vec{x},\tau) + g_I(\vec{x},\tau)\int d^3y\left(\Gamma^{IJ}[\psi;t,\vec{y},\vec{x})\,\chi_J(\vec{y},\tau) + \frac{\delta\Gamma^{IJ}[\psi;\tau,\vec{x},\vec{y})}{\delta\psi^J(\vec{y},\tau)}\right)\right],$$

$$S_0[\psi,\chi;t_0,t) = \int_{t_0}^{t} d\tau \int d^3x\left[\dot{\psi}^I(\vec{x},\tau)\,\chi_I(\vec{x},\tau) - \Lambda^I[\psi;\vec{x},\tau)\,\chi_I(\vec{x},\tau) - \frac{i}{2}\frac{\delta\Lambda^I[\psi;\vec{x},\tau)}{\delta\psi^I(\vec{x},\tau)}\right]$$

These χ are variables Fourier-conjugated to noises f in the latter's infinite-dimensional space. Integrating A in $[d\chi]$ is all but granted

Ensemble statistics on the (ψ,χ) space, normalization of A, definition of stochastic action $S[\psi,\chi;t_0,t)$, …

$$\begin{cases} \langle F \rangle = \int [d\psi] \int [d\chi] \, F[\psi] \, A[\psi, \chi; t_0, t) \, , \\ \\ \int [d\psi] \int [d\chi] \, A[\psi, \chi; t_0, t) = 1, \\ \\ A[\psi, \chi; t_0, t) = N_0 e^{-iS[\psi,\chi;t_0,t)} \end{cases}$$

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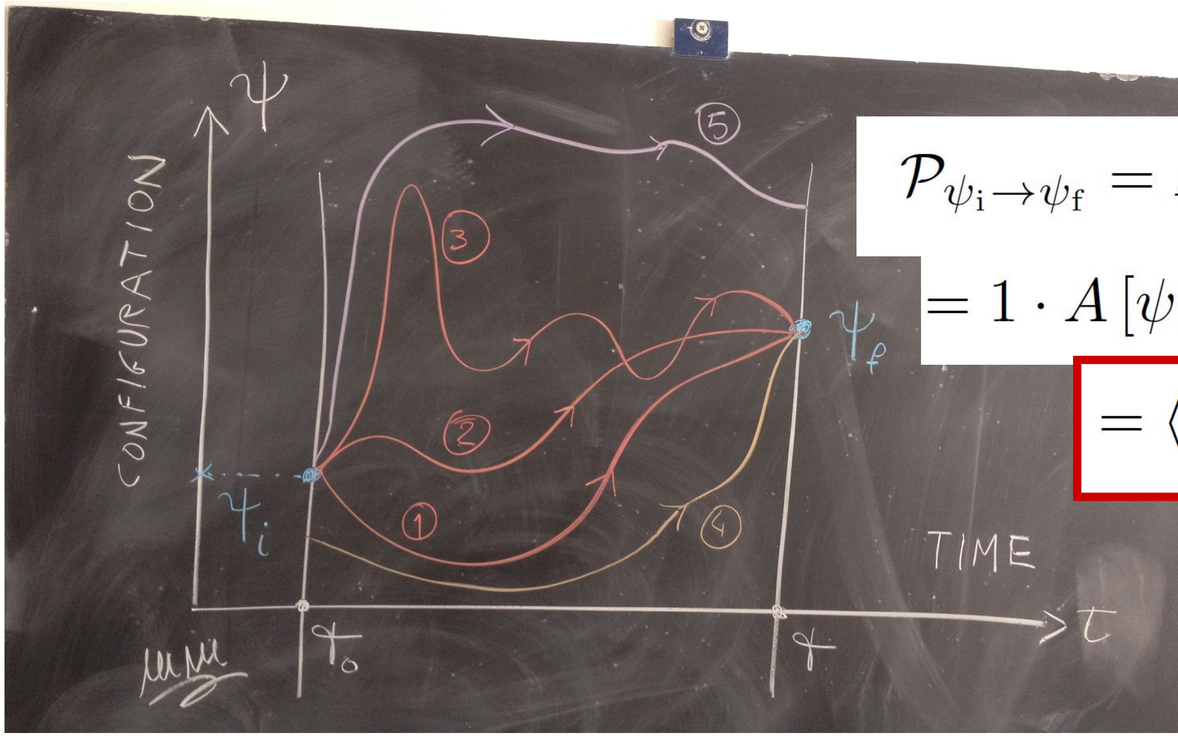
Transition probability between an initial field configuration ψ_i and a final one ψ_f:



$$\mathcal{P}_{\psi_\mathrm{i}\to\psi_\mathrm{f}} = A[\psi_1] + A[\psi_2] + A[\psi_3] =$$

$$= 1\cdot A[\psi_1] + 1\cdot A[\psi_2] + 1\cdot A[\psi_3] =$$

$$\boxed{= \langle 1 \rangle_{[\psi_1,\psi_2,\psi_3]}}$$

This is simply the ensemble average of "1" over all possible trajectories going from the selected initial condition to the selected final condition. Nothing changes when functional integrals are dealt with.

$$\mathcal{P}_{\psi_\mathrm{i}\to\psi_\mathrm{f}}(t_0,t) = \int\limits_{\substack{\psi(t_0)=\psi_\mathrm{i}\\ \psi(t)=\psi_\mathrm{f}}} [d\psi] \int [d\chi]\, A[\psi,\chi]$$

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In order to understand what goes on in a functional integral, go discrete:

$$\mathcal{P}_{\psi_{\mathrm{i}}\to\psi_{\mathrm{f}}}(t_0,t) = \lim_{\substack{N\to+\infty \\ \Delta t\to 0}} \left(\prod_{h=1}^{N-1}\int [d\psi(\tau_h)]\right)\left(\prod_{k=0}^{N}\int [d\chi(\tau_k)]\right) A[\psi,\chi;t_0,t)$$

Definition of a stochastic Lagrangian *L*, econding the whole statistical dynamics:

$$\mathcal{A}[\psi;t_0,t) = N_0(t_0,t)\exp\left(-i\int_{t_0}^{t} L[\psi;\tau)\,d\tau\right)$$

$$\mathcal{P}_{\psi_{\mathrm{i}}\to\psi_{\mathrm{f}}}(t_0,t) = \lim_{\substack{N\to+\infty \\ \Delta t\to 0}} N_0(t_0,t)\, e^{-\frac{i(t-t_0)}{N}L[\psi_{\mathrm{i}};t_0)} *$$

$$* \left(\prod_{h=1}^{N-1}\int [d\psi(\tau_h)]\right) e^{-\frac{i(t-t_0)}{N}L[\psi_{\mathrm{f}};t)} e^{-\frac{i(t-t_0)}{N}\sum_{k=1}^{N-1} L[\psi(\tau_k);\tau_k)}$$

Finite length time- or space-correlation for noises can give rise to time- or space-non-local stochastic Lagrangian *densities*, in a SFT

$$
\begin{cases}
A[\psi,\chi] = N_0 C[\chi,\Gamma]\, e^{-iS_0[\psi,\chi]} = N_0 e^{-iS_C[\chi,\Gamma]} e^{-iS_0[\psi,\chi]}, \\
S_0[\psi,\chi] = \int_{t_0}^{t} d\tau \int d^3x \mathcal{L}_0\left(\psi,\dot{\psi},\chi\right), \\
S_C[\psi,\chi] = \int_{t_0}^{t} d\tau \int d^3x \mathcal{L}_C(\psi,\chi), \\
S[\psi,\chi] = S_0[\psi,\chi] + S_C[\chi,\Gamma], \\
S[\psi,\chi] = \int_{t_0}^{t} d\tau \int d^3x \mathcal{L}\left(\psi,\dot{\psi},\chi\right), \\
\mathcal{L}\left(\psi,\dot{\psi},\chi\right) = \mathcal{L}_0\left(\psi,\dot{\psi},\chi\right) + \mathcal{L}_C(\psi,\chi)
\end{cases}
$$

The "silent Lagrangian density" is time-local (in plasmas, it is also space-local), i.e. has no memory effects. Every non-locality may come from the noisy addendum S_C, in which multiple time and point correlations of noises enter.

Let's try to convince the audience of this fact...

Calculation of S_C (1/3):

$$S_C\left[\chi,\Gamma\right] = i\ln\left\langle e^{-iS_{f,g}}\right\rangle_{f,g}, \quad S_{f,g} = \int_{t_0}^{t} d\tau \int d^3x \mathcal{L}_{f,g},$$

$$\mathcal{L}_{f,g} \stackrel{\text{def}}{=} -f^I\left(\vec{x},\tau\right)\chi_I\left(\vec{x},\tau\right) - g_I\left(\vec{x},\tau\right) R^I\left[\psi,\chi;\tau,\vec{x}\right),$$

$$R^I\left[\psi,\chi;\tau,\vec{x}\right) \stackrel{\text{def}}{=} \int d^3y \left(\Gamma^{IJ}\left[\psi;\tau,\vec{y},\vec{x}\right)\chi_J\left(\vec{y},\tau\right) + \frac{\delta\Gamma^{IJ}\left[\psi;\tau,\vec{x},\vec{y}\right)}{\delta\psi^J\left(\vec{y},\tau\right)}\right)$$

$$S_C\left[\chi,\Gamma;t_0,t\right) = i\ln\left\{1 + \sum_{n=1}^{+\infty}\frac{1}{i^n n!}\left\langle\left(S_{f,g}\left[f,g;\chi,\psi;t_0,t\right)\right)^n\right\rangle_{f,g}\right\},$$

$$\ln\left(1+x\right) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \mathbb{O}\left(x^5\right),$$

$$S_C^{(1)}\left[\chi,\Gamma\right] = \sum_{n=1}^{+\infty}\frac{1}{i^{n-1}n!}\left\langle\left(S_{f,g}\left[f,g;\chi,\psi\right]\right)^n\right\rangle_{f,g}$$

Calculation of S_C (2/3):

Calculation of S_C up to the first order in exp($S_{f,g}$)-1, and up to the second order in $S_{f,g}$: this is not necessarily an approximation, but just a dydactic truncation!

$$S_C^{(1)} = \int_{t_0}^{t} d\tau \int d^3x \mathcal{L}_C^{(1)}$$

$$\begin{cases} f_0^I(\vec{x},\tau) = \left\langle f^I(\vec{x},\tau)\right\rangle_{f,g}, \quad g_{0I}(\vec{x},\tau) = \left\langle g_I(\vec{x},\tau)\right\rangle_{f,g}, \\ \left(Q^{ff}\right)^{AB}(\vec{x},\vec{x}',\tau,\tau') = \left\langle f^I(\vec{x},\tau) f^{I'}(\vec{x}',\tau')\right\rangle_{f,g}, \\ \left(Q^{fg}\right)^I{}_{I'}(\vec{x},\vec{x}',\tau,\tau') = \left\langle f^I(\vec{x},\tau) g_{I'}(\vec{x}',\tau')\right\rangle_{f,g}, \\ Q^{gg}_{II'}(\vec{x},\vec{x}',\tau,\tau') = \left\langle g_I(\vec{x},\tau) g_{I'}(\vec{x}',\tau')\right\rangle_{f,g} \end{cases}$$

Calculation of S_C (3/3):

$$\mathcal{L}_C^{(1)}(\vec{x},\tau) = -f_0^I(\vec{x},\tau)\,\chi_I(\vec{x},\tau) - g_{0I}(\vec{x},\tau)\,R^I[\psi,\chi;\tau,\vec{x}) +$$

$$-\frac{i}{2}\chi_I(\vec{x},\tau)\int_{t_0}^{t} d\tau'\int d^3x'\left(Q^{ff}\right)^{IA}(\vec{x},\tau,\vec{x}',\tau')\,\chi_A(\vec{x}',\tau') +$$

$$-i\chi_I(\vec{x},\tau)\int_{t_0}^{t} d\tau'\int d^3x'\left(Q^{fg}\right)^{I}{}_{A}(\vec{x},\tau,\vec{x}',\tau')\,R^A[\psi,\chi;\tau',\vec{x}') +$$

$$-\frac{i}{2}R^I[\psi,\chi;\tau,\vec{x})\int_{t_0}^{t} d\tau'\int d^3x'\left(Q^{gg}\right)_{IA}(\vec{x},\tau,\vec{x}',\tau')\,R^A[\psi,\chi;\tau',\vec{x}') + \ldots$$

THANK YOU FOR YOUR ATTENTION, SEE YOU TOMORROW…

PLAN OF THE LECTURES

LECTURE 2/2

- **Application #1: stochastic field theory formulation of resistive MHD.**

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Nonlin. Processes Geophys., 15, 701–709, 2008
www.nonlin-processes-geophys.net/15/701/2008/

Nonlinear Processes in Geophysics

Turning the resistive MHD into a stochastic field theory

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MHD "field" equations of motion, with matter conservation and magnetic divergencelessness already included

$$\begin{cases} \partial_t B^i = B^j \partial_j V^i - B^i \partial_j V^j - V^j \partial_j B^i - \epsilon^{ijk} \partial_j \left(\zeta_{kh} J^h \right), \\ \\ \partial_t V^i = V^i \partial_j V^j - V^j \partial_j V^i + \dfrac{J_j}{\rho} B_k \epsilon^{jki} - \dfrac{\partial^i p}{\rho}, \\ \\ \zeta = \sigma^{-1} \end{cases}$$

$$\boxed{\begin{aligned} \partial_\tau \psi^I (\vec{x}, \tau) = \Lambda^I [\psi; \vec{x}, \tau) + f^I (\vec{x}, \tau) + \\ + \int g_J (\vec{y}, \tau) \Gamma^{JI} [\psi; \tau, \vec{y}, \vec{x}) \, d^3 y \end{aligned}}$$

OUR AGENDA IS:

1. **Recognizing those PDEs as a SFT;**
2. **Constructing the kernel *A*[*V*,*B*];**
3. **Giving a calculable example of *A*[*V*,*B*]**

$$\psi (\vec{x}, \tau) = \begin{pmatrix} \vec{B} (\vec{x}, \tau) \\ \vec{V} (\vec{x}, \tau) \end{pmatrix},$$

$$\chi (\vec{x}, \tau) = \begin{pmatrix} \vec{\Omega} (\vec{x}, \tau) \\ \vec{\Pi} (\vec{x}, \tau) \end{pmatrix}$$

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Give to some terms the **role of noises** and write MHD as a set of **Langevin field equations**

$$
\begin{cases}
\Xi^i \stackrel{\text{def}}{=} -\epsilon^{ijk}\partial_j\left(\zeta_{kh}J^h\right), \\
\Delta^i \stackrel{\text{def}}{=} \dfrac{J^i}{\rho}, \\
\Theta^i \stackrel{\text{def}}{=} -\dfrac{\partial^i p}{\rho}
\end{cases}
$$

In order for them to mimic the **granularity of matter**, possibly giving rise to out-of-the-(local)-equilibrium fluctuations, **noises** are chosen to depend on the **electric current**, the **matter density distribution** and **matter pressure**, J and ρ and p respectively

$$
\boxed{\vec{f}_B = \vec{\Xi}, \quad \vec{f}_V = \vec{\Theta}}
$$

Resistive, compressible MHD is turned into the following set of Langevin field equations:

$$
\begin{cases}
\partial_t B^i = B^j\partial_j V^i - B^i\partial_j V^j - V^j\partial_j B^i + \Xi^i, \\
\partial_t V^i = V^i\partial_j V^j - V^j\partial_j V^i + \Delta_j B_k \epsilon^{jki} + \Theta^i
\end{cases}
$$

Construct the stochastic action A[Ω,Π,*B*,*V*;t_0,*t*), as the product of a deterministic part and the noise factor *C*

$$A_0\left[\vec{\Omega},\vec{\Pi},\vec{B},\vec{V};t_0,t\right)\stackrel{\text{def}}{=}$$

$$A = N_0 C A_0$$

$$\stackrel{\text{def}}{=}\exp\left(-iS_0\left[\vec{\Omega},\vec{\Pi},\vec{B},\vec{V},\partial_t\vec{B},\partial_t\vec{V};t_0,t\right)\right)\stackrel{\partial}{=}$$

$$\stackrel{\partial}{=}\exp\left\{-i\int_{t_0}^{t}dt\int d^3x\left(\Omega_i\dot{B}^i+\Pi_i\dot{V}^i+\left(B^i\partial_jV^j+V^j\partial_jB^i-B^j\partial_jV^i\right)\Omega_i+\right.\right.$$

$$\left.\left.+\left(V^j\partial_jV^i-V^i\partial_jV^j\right)\Pi_i-\frac{5}{2}i\partial_iV^i\right)\right\}$$

$$C\left[\vec{\Omega},\vec{\Pi},\vec{B},\vec{V};t_0,t\right)\stackrel{\partial}{=}\left\langle\exp\left\{i\int_{t_0}^{t}dt\int d^3x\left[\Xi^i\left(\vec{x},\tau\right)\Omega_i\left(\vec{x},\tau\right)+\right.\right.\right.$$

$$\left.\left.\left.+\Theta^i\left(\vec{x},\tau\right)\Pi_i\left(\vec{x},\tau\right)+\epsilon^{ijk}\Delta_i\left(\vec{x},\tau\right)\Pi_j\left(\vec{x},\tau\right)B_k\left(\vec{x},\tau\right)\right]\right\}\right\rangle_{\vec{\Xi},\vec{\Delta},\vec{\Theta}}$$

Get rid of the ancillary variables Ω and Π, provided the noise statistics in the factor *C* is "kind enough" to let you make the integrals...

$$\mathcal{A}\left[\vec{B},\vec{V};t_0,t\right) =$$

$$= N_0\left(t_0,t\right)\int\left[d\vec{\Omega}\right]\int\left[d\vec{\Pi}\right] C\left[\vec{\Omega},\vec{\Pi},\vec{B},\vec{V};t_0,t\right) e^{-iS_0\left[\vec{\Omega},\vec{\Pi},\vec{B},\vec{V};t_0,t\right)}$$

Noise statistics are originated by the statistics of ρ, *p* and *J*, originated, in turn, by the microscopic behaviour of plasma particles

$$\begin{array}{ccc} \text{(complex)} & & \\ \text{particle dynamics} & \longmapsto & \mathcal{P}_{\mathrm{dyn}}\left[\rho,p,\vec{J}\right] \\ & & \swarrow \\ C\left[\vec{\Omega},\vec{\Pi},\vec{B},\vec{V};t_0,t\right) & \longleftarrow & \mathcal{Q}\left[\vec{\Xi},\vec{\Delta},\vec{\Theta}\right] \end{array}$$

Truncated Lagrangian density for the SMHD (1/2):

$$\left\langle \vec{J}\rho \right\rangle_{\vec{J},\rho,p} = 0, \quad \left\langle \vec{J}p \right\rangle_{\vec{J},\rho,p} = 0, \quad \langle \rho p \rangle_{\vec{J},\rho,p} = 0$$

$$\Downarrow$$

$$\left\langle \vec{\Xi} \otimes \vec{\Theta} \right\rangle_{\Xi,\Delta,\Theta} = 0, \; \left\langle \vec{\Xi} \otimes \vec{\Delta} \right\rangle_{\Xi,\Delta,\Theta} \neq 0, \; \left\langle \vec{\Theta} \otimes \vec{\Delta} \right\rangle_{\Xi,\Delta,\Theta} \neq 0$$

$$\boxed{\mathcal{L}^{(1)}_{\mathrm{SMHD}} = \mathcal{L}_{\mathrm{loc}} + \mathcal{L}^{(1)}_{\mathrm{non-loc}}}$$

$$\mathcal{L}_{\mathrm{loc}} \left(\vec{\Omega}, \vec{\Pi}, \partial_t \vec{B}, \partial_t \vec{V}, \vec{B}, \vec{V} \right) =$$

$$= \vec{\Omega} \cdot \partial_t \vec{B} + \vec{\Pi} \cdot \partial_t \vec{V} + \left[\left(\vec{\partial} \cdot \vec{V} \right) \vec{B} + \left(\vec{V} \cdot \vec{\partial} \right) \vec{B} - \left(\vec{B} \cdot \vec{\partial} \right) \vec{V} \right] \cdot \vec{\Omega} +$$

$$+ \left[\left(\vec{\partial} \cdot \vec{V} \right) \vec{V} - \left(\vec{V} \cdot \vec{\partial} \right) \vec{V} \right] \cdot \vec{\Pi} - \vec{\Xi}_0 \cdot \vec{\Omega} - \vec{\Theta}_0 \cdot \vec{\Pi} - \vec{\Delta}_0 \cdot \left(\vec{\Omega} \times \vec{B} \right)$$

Truncated Lagrangian density for the SMHD (2/2):

$$\mathcal{L}^{(1)}_{\mathrm{non-loc}}\left(\vec{\Omega},\vec{\Pi},\vec{B},\vec{V}\right) = -\frac{i}{2}\Omega_i(\vec{x},\tau)\int_{t_0}^{t} d\tau' \int d^3x' \left(Q_B^{\Xi\Xi}\right)^{ij}(\vec{x},\tau,\vec{x}',\tau')\,\Omega_j(\vec{x}',\tau') +$$

$$-\frac{i}{2}\Pi_i(\vec{x},\tau)\int_{t_0}^{t} d\tau' \int d^3x' \left(Q_V^{\Xi\Xi}\right)^{ij}(\vec{x},\tau,\vec{x}',\tau')\,\Pi_j(\vec{x}',\tau') +$$

$$-\frac{i}{2}\Omega_i(\vec{x},\tau)\int_{t_0}^{t} d\tau' \int d^3x' \left(Q_B^{\Theta\Theta}\right)^{ij}(\vec{x},\tau,\vec{x}',\tau')\,\Omega_j(\vec{x}',\tau') +$$

$$-\frac{i}{2}\Pi_i(\vec{x},\tau)\int_{t_0}^{t} d\tau' \int d^3x' \left(Q_V^{\Theta\Theta}\right)^{ij}(\vec{x},\tau,\vec{x}',\tau')\,\Pi_j(\vec{x}',\tau') +$$

$$\boxed{\left(Q^{\Theta\Xi}\right)^{IA} = 0,}$$

$$\boxed{Q^{\Xi\Xi} = \begin{pmatrix} Q_B^{\Xi\Xi} & 0 \\ 0 & Q_V^{\Xi\Xi} \end{pmatrix},}$$

$$\boxed{Q^{\Theta\Theta} = \begin{pmatrix} Q_B^{\Theta\Theta} & 0 \\ 0 & Q_V^{\Theta\Theta} \end{pmatrix}}$$

$$-i\epsilon^{jkh}\Pi_i(\vec{x},\tau)\int_{t_0}^{t} d\tau' \int d^3x' \left(Q_V^{\Xi\Delta}\right)^{i}{}_{j}(\vec{x},\tau,\vec{x}',\tau')\,\Omega_k(\vec{x}',\tau')\,B_h(\vec{x}',\tau') +$$

$$-i\epsilon^{jkh}\Pi_i(\vec{x},\tau)\int_{t_0}^{t} d\tau' \int d^3x' \left(Q_V^{\Theta\Delta}\right)^{i}{}_{j}(\vec{x},\tau,\vec{x}',\tau')\,\Omega_k(\vec{x}',\tau')\,B_h(\vec{x}',\tau') +$$

$$-\frac{i}{2}\epsilon^{i\ell m}\epsilon^{jhk}\Omega_\ell(\vec{x},\tau)\,B_m(\vec{x},\tau)\int_{t_0}^{t} d\tau' \int d^3x' \left(Q_V^{\Delta\Delta}\right)_{ij}(\vec{x},\tau,\vec{x}',\tau')\,\Omega_h(\vec{x}',\tau')\,B_k(\vec{x}',\tau')$$

Gaussian toy model of stochastic MHD: the noises are **delta-correlated Gaussian processes** depending on position and time

- **(Not trivially, but still) Calculable stochastic Lagrangian density**

$$\boxed{\mathcal{A}\left[\vec{B},\vec{V};t_0,t\right) = N_0\left(t_0,t\right) e^{-i\int_{t_0}^{t} d\tau \int d^3x\,\mathcal{L}\left[\vec{B},\vec{V},\partial_t\vec{B},\partial_t\vec{V};\tau\right)}}$$

$$\begin{cases} \Xi_0^i - \dot{B}^i - B^i\partial_j V^j - V^j\partial_j B^i + B^j\partial_j V^i \stackrel{\text{def}}{=} \varphi_B^i\left(\vec{B},\vec{V}\right), \\ \Theta_0^i - \epsilon^{k\ell i}B_k\Delta_{0\ell} - \dot{V}^i - V^j\partial_j V^i + V^i\partial_j V^j \stackrel{\text{def}}{=} \varphi_V^i\left(\vec{B},\vec{V}\right), \\ \left(a_\Theta^{-1}\right)^{\ell j} + \left(a_\Delta^{-1}\right)^{ab}\epsilon^{kj}{}_a\epsilon^{m\ell}{}_b B_k B_m \stackrel{\text{def}}{=} \mathcal{G}_{\Theta\Delta}^{\ell j}\left(\vec{B}\right) \end{cases}$$

Some useful definitions, just to shorten writing *L*

$$\mathcal{L}\left[\vec{B},\vec{V},\partial_t\vec{B},\partial_t\vec{V};\tau\right) = \mathcal{L}_{\text{field}}\left[\vec{B},\vec{V},\partial_t\vec{B},\partial_t\vec{V};\tau\right) + \lambda$$

$$\boxed{\mathcal{L}_{\text{field}} = -i\ln\left(\sqrt{\det\left\|\mathcal{G}_{\Theta\Delta}^{ij}\right\|}\right) - i\left[a_\Xi^{ij}\varphi_{Bi}\varphi_{Bj} + \left(\mathcal{G}_{\Theta\Delta}^{-1}\right)^{ij}\varphi_{Vi}\varphi_{Vj}\right],\quad \lambda = i\ln\left(64^3\pi^3\sqrt{\det\left\|a_\Xi^{ij}\right\|}\right)}$$

PLAN OF THE LECTURES

LECTURE 2/2

• **Application #1: stochastic field theory formulation of resistive MHD.**

• **Application #2: stochastic tetrad dynamics for turbulent MHD plasmas.**

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Lagrangian tetrad dynamics and the phenomenology of turbulence

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The stochastic tetrad magneto-hydrodynamics via functional formalism

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Turbulence is…

- **Erratic behaviour** of fluid dynamic proxies, **highly irregular** and **unpredictable**;

- **Noise… Langevin equations… Functional formalism…**

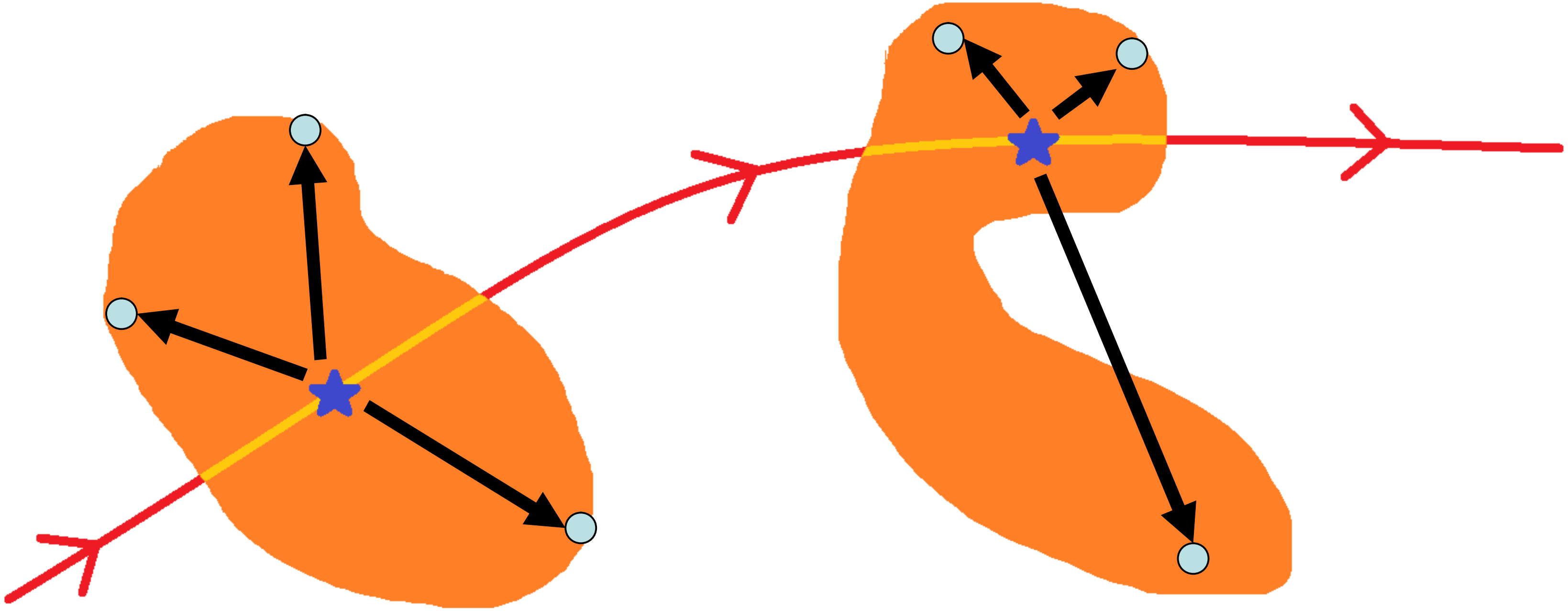
- **Big role palyed by granularity of matter, in general: important role of the material nature of fluids (and plasmas);**

- **Use material description of the plasma instead of the local one, i.e. *go Lagrangian instead of Eulerian*;**

- **Important multi-scale coupling, with cascades, eddy fragmentation and/or coalescence.**

- **Define scale-specific dynamical variables, so that their coupling will be the inter-scale coupling.**

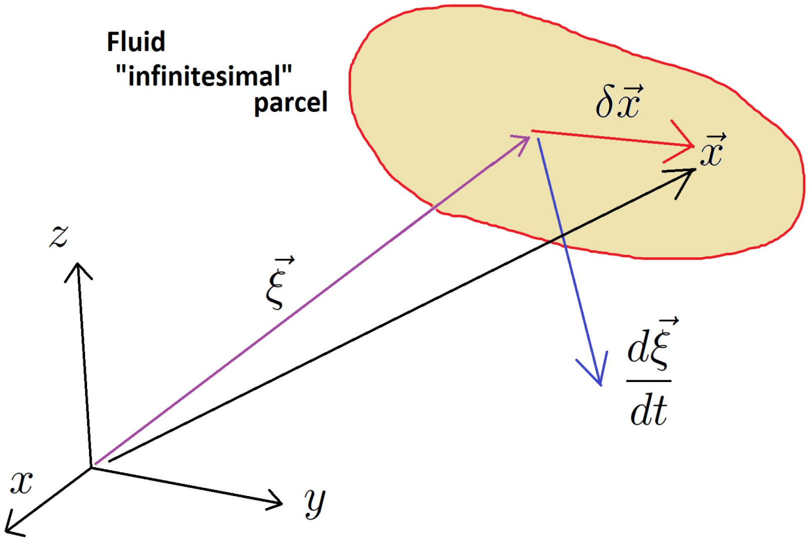
Representing the evolution of the material parcel of plasma



Once a reference point in the parcel is considered (e.g., sensibly the center-of-mass of the parcel), its "shape" is described by the position of three more material points.

Those three evolving relative positions form the tetrad

Tetrad variables and tetrad ODEs (1/6)



$$
\begin{cases}
A = \vec{\partial} \otimes \vec{V}, \quad Z = \vec{\partial} \otimes \vec{B}, \\
\mathrm{Tr}\,(A) = A^i{}_i = 0, \quad \mathrm{Tr}\,(Z) = Z^j{}_j = 0, \\
V^i\,(\vec{x}) = V^i\left(\vec{\xi}\right) + \partial_j V^i\left(\vec{\xi}\right)\delta x^j + \ldots, \\
B^i\,(\vec{x}) = B^i\left(\vec{\xi}\right) + \partial_j B^i\left(\vec{\xi}\right)\delta x^j + \ldots
\end{cases}
$$

Consider the relative velocity of the I-th point x_I with respect to the reference ξ: as long as the parcel is "infinitesimal" one can be happy with a linear approximation...

$$
\vec{\rho}_I = \vec{x}_I - \vec{\xi}, \qquad \frac{d\rho_I^i}{dt} = \rho_I^j A_j{}^i + o\,(\rho_I)\,, \quad \rho_I = |\delta\vec{x}_I|
$$

As the size of the parcel must play the role of physical scale, ONE NEEDS TO WORK WITH FINITE SIZE PARCELS

Massimo Materassi

Tetrad variables and tetrad ODEs (2/6)

Stochastic tetrad assumption: the difference of material velocities between two points of the fluid ρ-apart from each other exceeds a purely ρ-linear approximation of a stochastic process, the average of which is o(ρ)

$$\frac{d\rho_I^i}{dt} = \rho_I^j M_j{}^i + u_I^i,$$

$$\langle u_I^i \rangle = o\left(\rho_I\right)$$

$$\begin{cases} \varrho^{jI} = \rho_I^j, \quad U^{jI} = u_I^j, \\ \dot{\varrho} = \varrho^\dagger \cdot M + U, \\ \kappa = \varrho^{-1}, \quad \Pi = \dfrac{\kappa \cdot \kappa^\dagger}{\mathrm{Tr}\left(\kappa \cdot \kappa^\dagger\right)} \end{cases}$$

Tetrad matrix variables: ρ represents the tetrad configuration variables, *M* is the coarse grained velocity gradient, while Π is the tetrad shape tensor

$$\Pi\left(\hat{i}, \hat{j}, \hat{k}\right) = \mathbf{1}$$

Tetrad variables and tetrad ODEs (3/6)

The Lagrangian (material) dynamics of the incompressible, visco-resistive plasma is the starting point:

$$\begin{cases} \dot{V}^i = -\dfrac{1}{2\rho}\partial^i B^2 + \dfrac{1}{\rho}B^j\partial_j B^i - \dfrac{1}{\rho}\partial^i p + \dfrac{1}{\rho}\partial_j\sigma^{ij}, \\ \\ \dot{B}^i = B^j\partial_j V^i + \zeta\partial^2 B^i, \\ \\ \sigma^{ij} = \left[\eta\left(\delta^{ni}\delta^{mj} + \delta^{nj}\delta^{mi} - \dfrac{2}{3}\delta^{ij}\delta^{mn}\right) + \nu\delta^{ij}\delta^{mn}\right]\partial_m V_n \stackrel{\text{def}}{=} \Sigma^{ijmn}A_{mn}. \end{cases}$$

Tetrad variables and tetrad ODEs (3/5)

Need of material equations for gradients *A* and *Z*:

$$
\begin{cases}
\dot{A} + A^2 - \left[\mathrm{Tr}\left(A^2\right) - \dfrac{\mathrm{Tr}\left(\Xi^2\right)}{2\rho}\right]\dfrac{\mathbf{1}}{3} - \dfrac{\Xi \cdot Z}{\rho} = \mathcal{H}, \\
\dot{Z} + [Z, A] = \Theta, \\
\Xi = Z - Z^\dagger, \\
\mathcal{H} = \dfrac{\eta}{\rho}\partial^2 A - \dfrac{1}{\rho}\left(\vec{\partial}\otimes\vec{\partial} - \dfrac{\mathbf{1}}{3}\partial^2\right)\left(p + \dfrac{B^2}{2}\right) + \dfrac{1}{\rho}\vec{B}\cdot\vec{\partial}Z, \\
\Theta = \zeta\partial^2 Z + \vec{B}\cdot\vec{\partial}A
\end{cases}
$$

- ***H*** **and Θ are gradients of gradients, i.e. have stronger irregularities than *A* and *Z*: those will become noises**

Tetrad variables and tetrad ODEs (4/5)

From infintiesimal parcels to tetrads: passing from ***A*** and ***Z*** to ***M*** and ***W***, i.e. coarse graining of the Lagrangian variable ODEs of incompressible, resistive MHD

$$\begin{cases} \dot{\varrho} = \varrho^{\dagger} \cdot M + U, \\ \dot{M} + M^2 - \left[\mathrm{Tr}\left(M^2\right) - \dfrac{\mathrm{Tr}\left(\Xi^2\right)}{2\rho} \right] \Pi - \dfrac{\Xi \cdot W}{\rho} = \mathcal{H}, \\ \dot{W} + [W, M] = \Theta \end{cases}$$

- **Noise *H* for the ODE of *M*: save incompressiblity, behave correctly with respect to the K41 turbulence theory, do this all in the minimal way…**

- **Noise Θ for the ODE of *W*: do the same! ☺**

$$\begin{cases} \mathcal{H} = h + \alpha_V \left[M^2 - \left(\mathrm{Tr}\left(M^2\right) - \dfrac{\mathrm{Tr}\left(\Xi^2\right)}{2\rho} \right) \Pi - \dfrac{\Xi \cdot Z}{\rho} \right], \\ \Theta = \theta + \alpha_B \left[Z, A \right] \end{cases}$$

$$\begin{cases} \dot{\varrho} = \varrho^{\dagger} \cdot M + U, \\ \dot{M} = (\alpha_V - 1) \left\{ M^2 - \left[\mathrm{Tr}\left(M^2\right) - \frac{\mathrm{Tr}\left(\Xi^2\right)}{2\rho} \right] \Pi - \frac{\Xi \cdot W}{\rho} \right\} + h, \\ \dot{W} = (\alpha_B - 1) \left[W, M \right] + \theta \end{cases}$$

- **Langevin Equations for Tetrad MHD show purely additive noises (*U*,*h*,θ).**

- **These are the same type of ODEs of finite dimensional SDS with pure additive noises, as Phythian's:**

$$\dot{\psi} = \Lambda\left(\psi\right) + f$$

Stochastic Tetrad MHD (1/4)

All the noises are supposed **Gaussian distributed and delta-correlated in time**: this will allow directly for the formulation of **the stochastic kernel in terms of physical variables**.

Moreover, their **average will be assumed to be zero**, coherently with the assumption to be an **o-micron of the tetrad size**

$$\left\{\begin{aligned}
&\left\langle u_I^i(\tau)\, u_J^j(\tau')\right\rangle = \Sigma^{(u)}{}^{ij}_{IJ}(\tau)\,\delta(\tau-\tau')\,,\\
&\left\langle h^{ij}(\tau)\, h^{kh}(\tau')\right\rangle = \Sigma^{(h)ijkh}(\tau)\,\delta(\tau-\tau')\,,\\
&\left\langle \theta^{ij}(\tau)\, \theta^{kh}(\tau')\right\rangle = \Sigma^{(\theta)ijkh}(\tau)\,\delta(\tau-\tau')\,,\\
&\Sigma^{(u)}{}^{ij}_{IJ} = 2\sqrt{\boldsymbol{M}^2}\left[C_{//}\rho_I^i\rho_J^j + C_\perp\left(\boldsymbol{\rho}^2\delta^{ij}\delta_{IJ} - \rho_I^i\rho_J^j\right)\right],\\
&\Sigma^{(h)abcd} = \frac{2C_h}{\boldsymbol{\rho}^2}\left(\delta^{ad}\delta^{bc} - \frac{1}{3}\delta^{ab}\delta^{cd}\right),\\
&\Sigma^{(\theta)abcd} = \frac{2C_\theta}{\boldsymbol{\rho}^2}\left(\delta^{ad}\delta^{bc} - \frac{1}{3}\delta^{ab}\delta^{cd}\right) \qquad \boldsymbol{F}^2 = \mathrm{Tr}\left(F^\dagger F\right)
\end{aligned}\right.$$

Massimo Materassi

Stochastic Tetrad MHD (2/4)

$$\psi = \begin{pmatrix} \varrho \\ M \\ W \end{pmatrix}$$

Tetrad dynamical variables are the elements of the configuration matrix ρ, of the coarse-grained traceless *V*-gradient *M* and of the coarse-grained traceless *B*-gradient *W*.

$$\mathcal{A}\left[\varrho, M, W\right] = \int \left[d\chi_{\varrho}\right] \int \left[d\chi_{M}\right] \int \left[d\chi_{W}\right] A\left[\varrho, \chi_{\varrho}, M, \chi_{M}, W, \chi_{W}\right]$$

$$\begin{cases} \mathcal{Q}\left(f\left(\tau\right)\right) = \dfrac{e^{-\frac{1}{2}\int_{t_0}^{t} d\tau' \sum_{\alpha,\beta}^{-1}(\tau)\delta(\tau-\tau')\left(f^{\alpha}(\tau)-f_0^{\alpha}(\tau)\right)\left(f^{\beta}(\tau')-f_0^{\beta}(\tau')\right)}}{\sqrt{\det(2\pi\Sigma)}}, \\ \left\langle f^{\alpha}\left(\tau\right) f^{\beta}\left(\tau'\right)\right\rangle_f = \Sigma^{\alpha\beta}\left(\tau\right)\delta\left(\tau-\tau'\right), \\ C\left[\chi\right] = \left\langle e^{i\int_{t_0}^{t} d\tau \chi_{\alpha}(\tau) f^{\alpha}(\tau)}\right\rangle_f, \\ C\left[\chi\right] = e^{\frac{1}{2}\int_{t_0}^{t} d\tau\left(2i\underline{f}_0^{\dagger}\cdot\underline{\chi}-\underline{\chi}^{\dagger}\cdot\Sigma\cdot\underline{\chi}\right)} \end{cases}$$

The purely additive, Gaussian, delta-correlated noise in an SDS yields an "easy" calculation of the C factor for the stochastic kernel A[ψ,χ]

Stochastic Tetrad MHD (3/4)

$$\mathcal{A}\left[\varrho, M, W; t_0, t\right) = N_0\left(t_0, t\right) e^{-iS\left[\varrho, M, W; t_0, t\right)}$$

$$S\left[\varrho, M, W; t_0, t\right) = -\frac{i}{2}\int_{t_0}^{t} d\tau \left\{ \frac{\mathrm{Tr}\left[\left(\dot{\varrho} - \varrho^{\dagger}\cdot M\right)^{\dagger}\cdot\left(C_{//}^{-1} r + C_{\perp}\left(I - r\right)\right)\cdot\left(\dot{\varrho} - \varrho^{\dagger}\cdot M\right)\right]}{2\boldsymbol{\rho}^2\sqrt{\boldsymbol{M}^2}} + \right.$$

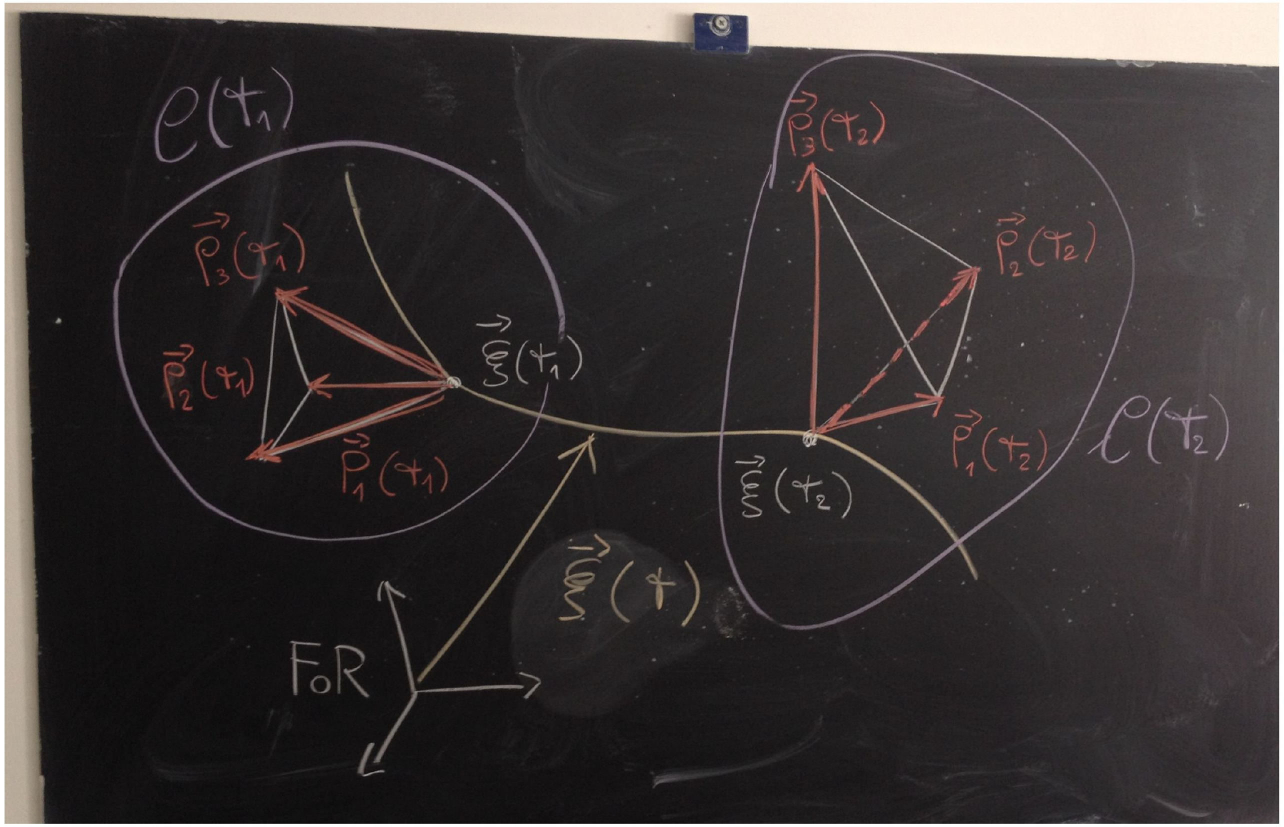
$$+\frac{\boldsymbol{\rho}^2}{2C_h}\left\|\dot{M} - \left(\alpha_v - 1\right)\left[M^2 - \left(\boldsymbol{M}^2 - \frac{\Xi^2}{2\rho}\right)\Pi\left(\boldsymbol{\rho}\right) - \frac{\Xi\cdot W}{\rho}\right]\right\|^2 +$$

$$\left. +\frac{\boldsymbol{\rho}^2}{2C_\theta}\left\|\dot{W} - \left(\alpha_B - 1\right)\left[W, M\right]\right\|^2 - 2\left(\alpha_v - 1\right)\mathrm{Tr}\left(\Pi\left(\boldsymbol{\rho}\right)\cdot M\right)\right\}$$

$$\boxed{\left\|F\right\|^2 = \mathrm{Tr}\left(F^{\dagger}F\right),\; r_{ab}^{IJ} = \hat{\rho}_a^I\hat{\rho}_b^J,\quad I_{ab}^{IJ} = \delta_{ab}\delta^{IJ}}$$

Stochastic Tetrad MHD (4/4)

$$\mathcal{P}_{\psi_\mathrm{i} \to \psi_\mathrm{f}}(t_0, t) = \int_{(\varrho_\mathrm{i}, M_\mathrm{i}, W_\mathrm{i})}^{(\varrho_\mathrm{f}, M_\mathrm{f}, W_\mathrm{f})} [d\varrho]\,[dM]\,[dW]\,\mathcal{A}[\varrho, M, W; t_0, t]$$



Conditioning the average $\langle 1 \rangle_{\rho, M, W}$ on different integration domains one may calculate the transition probabilities between different initial-final condition combinations (scale transitions, magnetic topology transitions etc...)

Note: plasma is infintiely many tetrads!

PLAN OF THE LECTURES

LECTURE 2/2

• Application #1 (2/2): stochastic field theory formulation of resistive MHD.

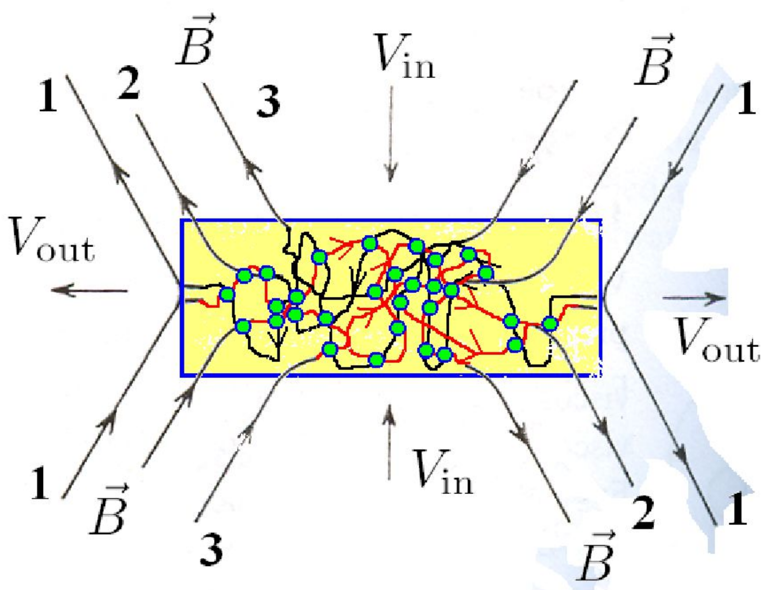
• Application #2: stochastic tetrad dynamics for turbulent MHD plasmas.

• Concluding remarks: invitation to curiosity.

Invitation to curiosity (1/2):
fractal 2d reconnection (I/III)

"Complexity and Turbulence in Space Plasmas", 18-22 September 2017, L'Aquila (Italy)

Invitation to curiosity (1/2):
fractal 2d reconnection (II/III)



The ratio between the incoming and outgoing plasma velocity may be calculated via the reconnected-mass conservation.

$$\mathcal{M} = \frac{V_{\rm in}}{V_{\rm out}}$$

$$\int_{I_x} V_y(x)\, d^{D_{\rm in}} \mu_{I_x} = \int_{I_y} V_x(y)\, d^{D_{\rm out}} \mu_{I_y}$$

$$\boxed{\mathcal{M}_{FRM} = \left(\frac{\sqrt{\pi}}{\ell_0}\right)^{D_{\rm out}-D_{\rm in}} \frac{D_{\rm in}\Gamma\left(\frac{D_{\rm in}}{2}\right)}{D_{\rm out}\Gamma\left(\frac{D_{\rm out}}{2}\right)} \frac{\delta^{D_{\rm out}}}{L^{D_{\rm in}}}}$$

Let the Hurst dimensions of the reconnected in- and out-front be $D_{\rm in}$ and $D_{\rm out}$, both in (0,1)

Invitation to curiosity (1/2):
fractal 2d reconnection (III/III)

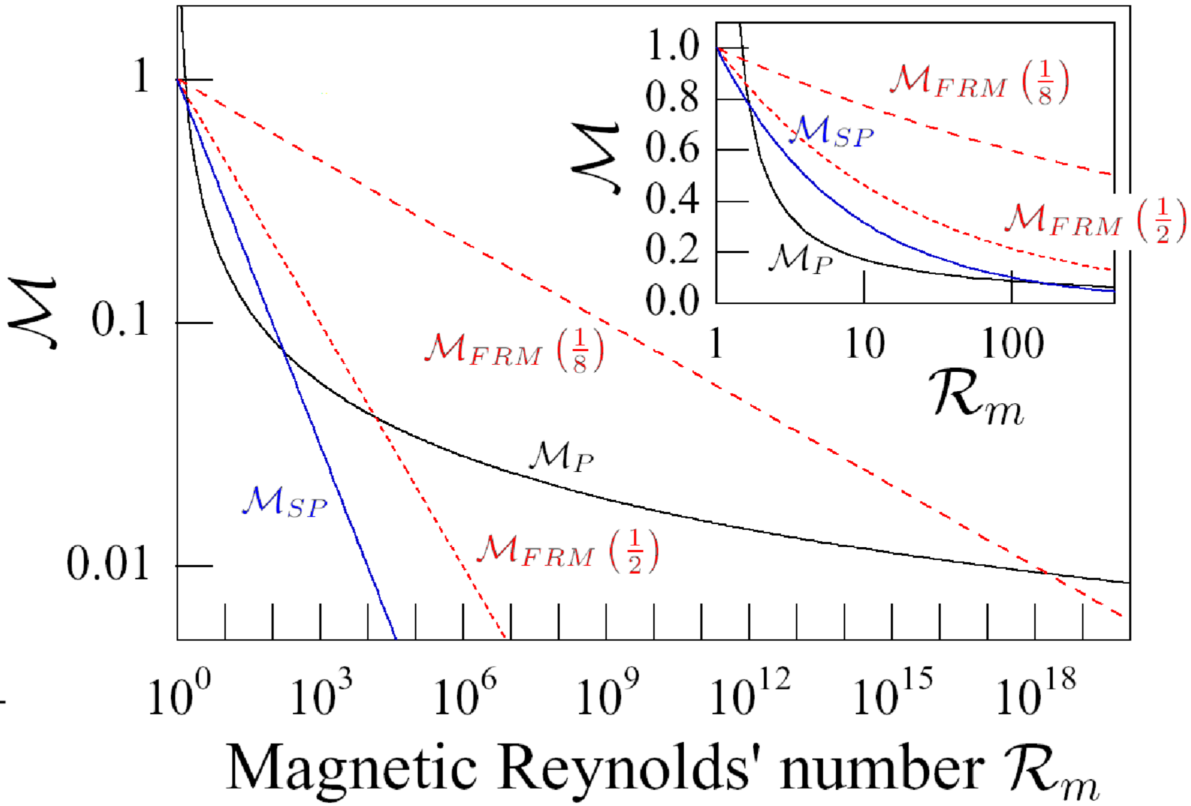
Dimensionally isotropic scenario:

$$D_{\mathrm{out}} = D_{\mathrm{in}} \stackrel{\mathrm{def}}{=} D$$

$$\mathcal{M}_{SP}(\mathcal{R}_m) = \frac{1}{\mathcal{R}_m^{\frac{1}{2}}}$$

$$\mathcal{M}_{P}(\mathcal{R}_m) = \frac{\pi}{8 \ln \mathcal{R}_m}$$

$$\mathcal{M}_{FRM}(\mathcal{R}_m, D) = \frac{1}{\mathcal{R}_m^{\frac{D}{D+1}}}$$



Materassi, M., Consolini, G., "Magnetic reconnection rate in space plasmas: a fractal approach", (2007) Physical Review Letters, 99 (17), art. no. 175002.

Invitation to curiosity (2/2): metriplectic dynamics in dissipative plasmas (I/III)

"Dissipation" making the system relax is represented by the collisional integral

Boltzmann equations with collisions:

$$
\left\{
\begin{aligned}
& z = (\vec{x}, \vec{v}) \, , \\
& \partial_t f = -\vec{v} \cdot \partial_{\vec{x}} f + \partial_{\vec{x}} \phi \left(\vec{x}, f \right] \cdot \partial_{\vec{v}} f + \\
& + \frac{\partial}{\partial v^i} \int dz' \omega^{ij} \left(z, z' \right) \left[\frac{\partial f(z)}{\partial v^j} f(z') - \frac{\partial f(z')}{\partial v'^j} f(z) \right] , \\
& \phi \left(\vec{x}, f \right] = \int dz' f(z') V(\vec{x}, \vec{x}') \, , \\
& \omega^{ij} = \omega^{ji}, \quad \omega^{ij}(z, z') = \omega^{ij}(z', z) \, , \quad \left(v_j - v'_j \right) \omega^{ij} = 0
\end{aligned}
\right.
$$

Invitation to curiosity (2/2): metriplectic dynamics in dissipative plasmas (II/III)

$$\begin{cases} \omega^{ij}(z, z') \to 0 : \\ \partial_t f = -\vec{v} \cdot \partial_{\vec{x}} f + \partial_{\vec{x}} \phi(\vec{x}, f] \cdot \partial_{\vec{v}} f, \\ \\ H[f] = \frac{1}{2} \int dz f(z) \left[v^2 + \phi(\vec{x}, f] \right], \\ \{A, B\} = \int dz \left[\partial_{\vec{x}} \left(\frac{\delta A}{\delta f(z)} \right) \cdot \partial_{\vec{v}} \left(\frac{\delta B}{\delta f(z)} \right) - \partial_{\vec{x}} \left(\frac{\delta B}{\delta f(z)} \right) \cdot \partial_{\vec{v}} \left(\frac{\delta A}{\delta f(z)} \right) \right], \\ \\ \partial_t f(z, t) = \{f(z, t), H[f]\} \end{cases}$$

In the non-collisional limit the system becomes Hamiltonian:

Shannon entropy of *f* is constant in the non-collisional limit:

$$\begin{cases} S[f] = -\int dz f(z) \ln f(z), \\ \{S, A\} = 0 \quad \forall \quad A, \\ \dot{S} = 0 \end{cases}$$

Invitation to curiosity (2/2): metriplectic dynamics in collisional plasmas (III/III)

$$\begin{cases} \langle\langle f(z), g(z)\rangle\rangle = \{f(z), g(z)\} + (f(z), g(z)), \\ (H, A) = 0, \quad \{S, A\} = 0, \quad (A, A) \geq 0 \quad \forall \quad A, \\ F[f] = H[f] + \alpha S[f], \quad \dot{H} = 0, \quad \dot{S} \geq 0, \\ \partial_t f(z) = \langle\langle f(z), F[f]\rangle\rangle \end{cases}$$

In the presence of collision the system is metriplectic and complete, i.e. has a generalized bracket taking into account of dissipation, that produces collisions

$$(A[f], B[f]) =$$

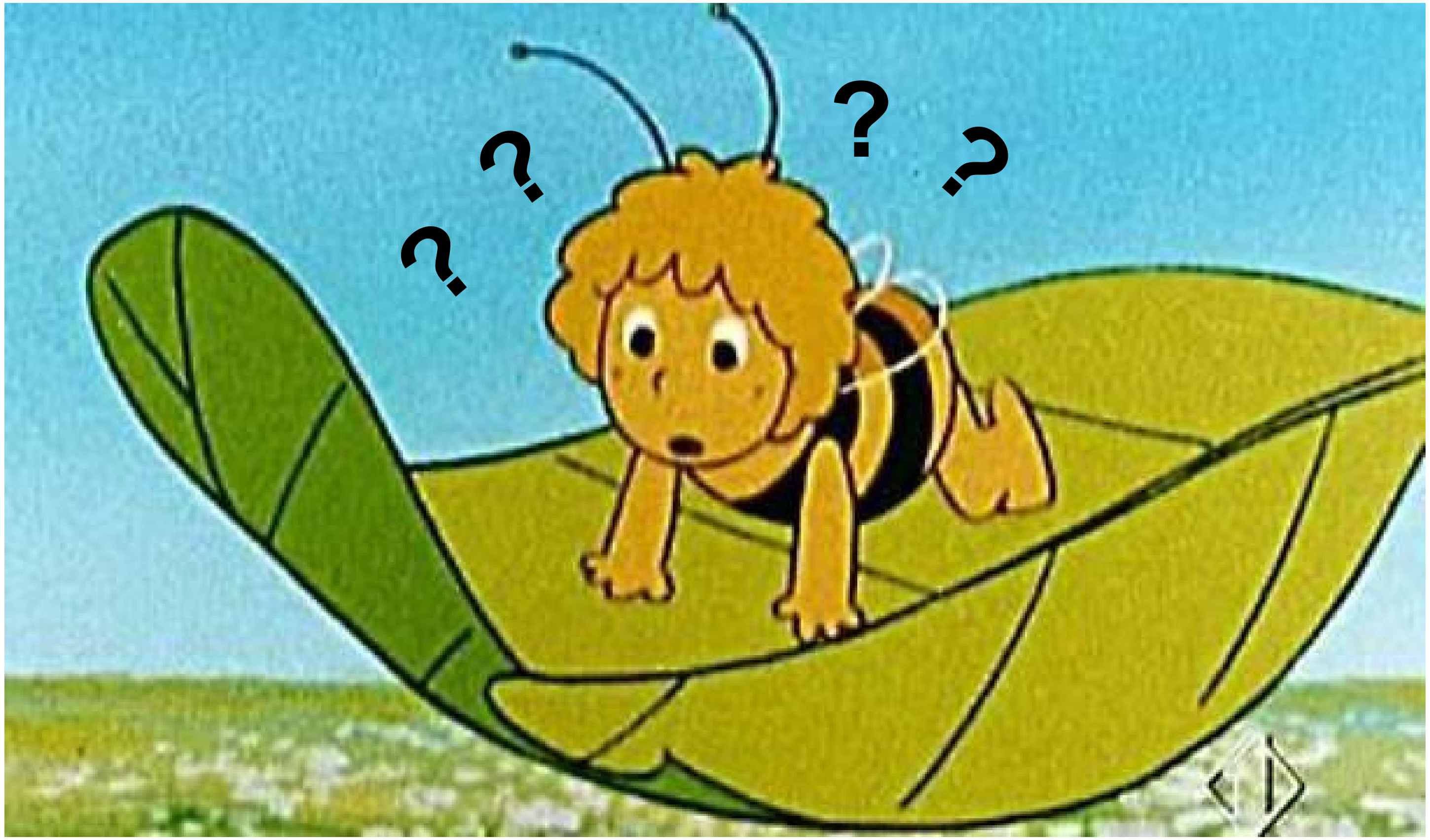
$$= \int dz \int dz' \frac{\omega^{ij}(z,z')}{2\alpha} f(z) f(z') \left[\frac{\partial}{\partial v^i}\left(\frac{\delta A}{\delta f(z)}\right) - \frac{\partial}{\partial v'^i}\left(\frac{\delta A}{\delta f(z')}\right)\right]\left[\frac{\partial}{\partial v^i}\left(\frac{\delta B}{\delta f(z)}\right) - \frac{\partial}{\partial v'^i}\left(\frac{\delta B}{\delta f(z')}\right)\right]$$

It is right Shannon entropy of the distribution *f* that generates the dissipative component of the motion (i.e. the collisional integral):

$$\alpha(f, S[f]) = \frac{\partial}{\partial v^i} \int dz' \omega^{ij}(z, z') \left[\frac{\partial f(z)}{\partial v^j} f(z') - \frac{\partial f(z')}{\partial v'^j} f(z)\right]$$

GENERAL CONCLUSION/SUGGESTION:

BE(E) CURIOUS!



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