Kappa Distributions: Turbulent Equilibria

1. Foundation: Plasma Kinetic Theory

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Kappa Distribution as Tsallis Equilibrium

Boltzmann-Gibbs definition of entropy

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i,$$

where $k = k_B = 1.3806503 \times 10^{-23} \mathrm{m}^2 \mathrm{kg\ s}^{-2} \mathrm{K}^{-1}$ for thermostatistics and $k = 1$ for information system (Shannon entropy). p_i is the probability of the system being in a particular state called the i-th state. W is the total number of possible states.

Kappa Distribution as Tsallis Equilibrium

Since p_i is the probability, their sum must be unity,

$$\sum_{i=1}^{W} p_i = 1.$$

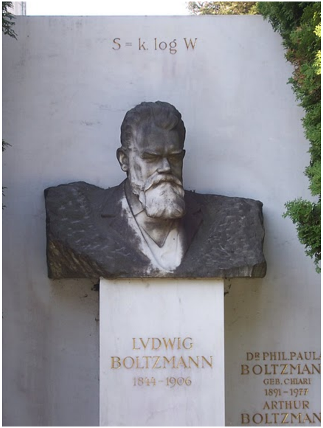
If the system has equal probability of being in any state, i,

$$p_i = \frac{1}{W},$$

and the entropy becomes

$$S_{BG} = -k \sum_{i=1}^{W} \frac{1}{W} \ln \frac{1}{W} = k \ln W.$$

Kappa Distribution as Tsallis Equilibrium



Kappa Distribution as Tsallis Equilibrium

Boltzmann entropy is extensive: Entropy of total system is sum of entropies of subsystems.

$$S_{BG}(A+B) = k\ln(W_A W_B) = k\ln W_A + k\ln W_B = S_{BG}(A) + S_{BG}(B).$$

The extensivity of the entropy is applicable for ideal gas and systems interacting with short-range force, as Boltzmann himself noted.

Kappa Distribution as Tsallis Equilibrium

"The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts" "Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled" [Enrico Fermi. *Thermodynamics* (Dover, New York, 1936), p. 53]

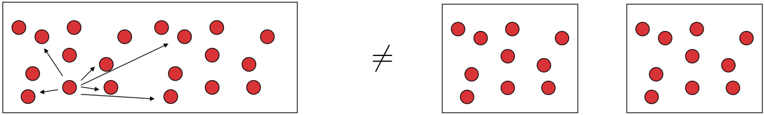
For *complex* systems or systems interacting with long-range force, such as gravitational bodies or plasmas, the *non-extensive* entropic principle may govern the statistical properties.

$$S(A+B) \neq S(A) + S(B).$$

The question is, "What is the appropriate mathematical form of non-extensive entropy?"

Kappa Distribution as Tsallis Equilibrium

$$S(A+B) \neq S(A) + S(B).$$



A possible answer was put forth by Tsallis [1988], but it was actually suggested in the information theory first. Tsallis simply rediscovered it independently,

$$S_q = k\,\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}, \qquad (q \to 1 \text{ equivalent to } S_{BG}).$$

Tsallis entropy has the non-additive, or *non-extensive* property,

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\,\frac{S_q(A)}{k}\frac{S_q(B)}{k}.$$

q is a parameter that determines the deviation from BG statistics, $q=1$ being the BG limit.

In the literature q is usually treated phenomenologically.

For electrons in unmagnetized plasma interacting with long-range Coulomb force, however, q can be calculated rigorously as

$$q = \frac{5}{9}$$

Kappa Distribution as Tsallis Equilibrium

For continuous system such as plasmas the Boltzmann-Gibbs entropy is

$$S_{BG} = -k_B\,\beta \int_{\mathscr{V}} d\mathbf{x} \int d\mathbf{v}\, \frac{f(\mathbf{x},\mathbf{v})}{\beta} \ln \frac{f(\mathbf{x},\mathbf{v})}{\beta}.$$

Equilibrium distribution f can be found by considering the Helmholtz free energy,

$$\begin{aligned} F &= U - TS, \\ U &= \int_{\mathcal{V}} d\mathbf{x} \int d\mathbf{v}\, \frac{mv^2}{2} f(\mathbf{x},\mathbf{v}). \end{aligned}$$

Kappa Distribution as Tsallis Equilibrium

Taking the functional derivative and setting equal to zero,

$$\frac{\delta F}{\delta f} = 0,$$

one obtains the Maxwell-Boltzmann-Gauss distribution

$$\boxed{f_{\rm M}(\mathbf{v}) = \frac{\beta}{e} \exp\left(-\frac{mv^2}{2k_B T}\right).}$$

Kappa Distribution as Tsallis Equilibrium

The continuous version of the Tsallis entropy is

$$S_q = -\frac{k_B\,\beta}{1-q}\int_{\mathcal{V}} d\mathbf{x}\int d\mathbf{v}\left[\frac{f(\mathbf{x},\mathbf{v})}{\beta}-\left(\frac{f(\mathbf{x},\mathbf{v})}{\beta}\right)^q\right].$$

Extremization of the Helmholtz free energy leads to

$$f_q(\mathbf{v}) = \frac{\beta\, q^{1/(1-q)}}{[1+(1-q)\,mv^2/2k_B T]^{1/(1-q)}}.$$

Kappa Distribution as Tsallis Equilibrium

Upon defining

$$\kappa \equiv \frac{1}{1-q}, \qquad q = \frac{\kappa - 1}{\kappa},$$

we obtain

$$f_\kappa(\mathbf{v}) \propto \frac{1}{[1 + mv^2/2\kappa\, k_B T]^\kappa}.$$

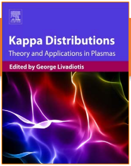
Kappa Distribution as Tsallis Equilibrium

A slightly different form can be obtained on the basis of "escort entropy" - See Livadiotis' book (2017) - the Vasyliunas-Olbert (kappa) distribution,

$$f_\kappa(\mathbf{v}) \propto \frac{1}{[1 + mv^2/2\kappa\, k_B T]^{\kappa+1}}.$$

For plasma electrons, I will show that the kappa value is

$$\kappa = \frac{9}{4}$$



Kappa Distributions

Theory and Applications in Plasmas

George Livadiotis, Senior Research Scientist, Southwest Research Institute, USA

ELSEVIER

Presents the theoretical developments of kappa distributions, their applications in geophysical, space, and astrophysical plasmas, and our understanding of statistical mechanics in plasmas and other particle systems residing in stationary states out of thermal equilibrium

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AUDIENCE

Kappa Distributions is ideal for space, plasma, and statistical physicists; geophysicists, especially of the upper atmosphere, Earth and planetary scientists, and astrophysicists

KEY FEATURES

- Answers important questions, such as how plasma waves are affected by kappa distributions and how solar wind, magnetospheres, and other geophysical, space, and astrophysical plasmas can be modeled using kappa distributions
- Presents the features of kappa distributions in the context of plasmas, including how kappa indices, temperatures, and densities vary among the species populations in different plasmas
- Provides readers with the information they need to decide which specific formula of kappa distribution should be used for a certain occasion and system (toolbox)

Kappa Distributions: Theory and Applications in Plasmas presents the theoretical developments of kappa distributions, their applications in plasmas, and how they affect the underpinnings of our understanding of space, plasma, and statistical physics, astrophysics, and thermodynamics. Separated into three major parts, the book covers theoretical methods, analytical methods in plasmas, and applications in space plasmas. The first part of the book focuses on basic aspects of the statistical theory of kappa distributions, beginning with their connection to the solid backgrounds of non-extensive statistical mechanics. The book then moves on to plasma physics, and is devoted to analytical methods related to kappa distributions on various basic plasma topics, spanning linear/nonlinear plasma waves, solitons, shockwaves, and dusty plasmas. The final part of the book deals with applications in space plasmas, focusing on applications of theoretical and analytical developments in space plasmas from the heliosphere and beyond, in other astrophysical plasmas.

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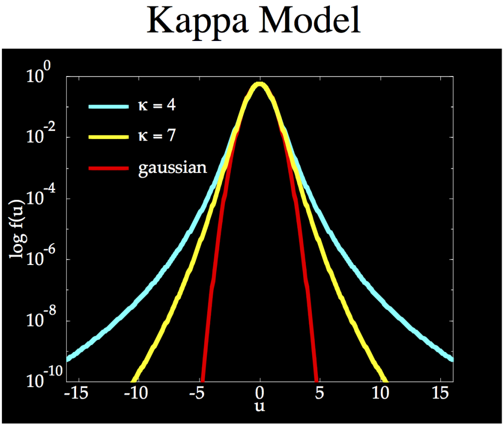
Gaussian vs Kappa Distribution

$$f_{gaussian}(u) = \frac{1}{\pi^{1/2}} \exp(-u^2)$$

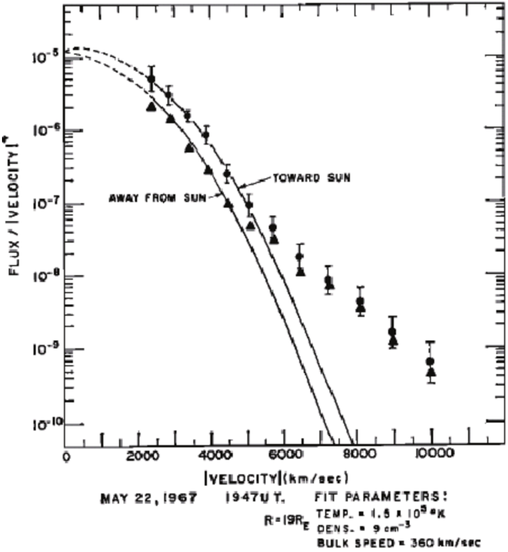
$$f_{kappa}(u) = \frac{1}{\pi^{1/2}} \frac{\Gamma(\kappa + 1)}{\kappa^{1/2}\Gamma(\kappa + 1/2)} \frac{1}{(1 + u^2/\kappa)^{\kappa+1}}$$

Kappa Distribution as Tsallis Equilibrium

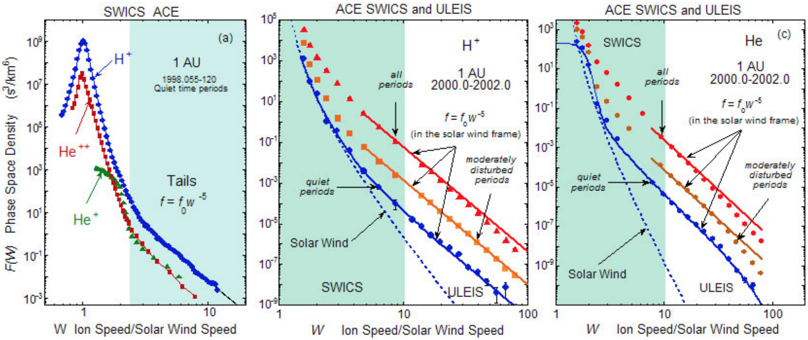


Kappa Distribution as Tsallis Equilibrium

Fig-1-Empirical-distribution-functions-of-the-solar-wind-electrons-measured-at-1-AU.png 850×921 pixels 9/18/17, 7:23 AM



Kappa Distribution as Tsallis Equilibrium



Fundamentals of Plasma

Kinetic Theory

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- A I Akhiezer, et al., *Plasma Electrodynamics, Vol. 2* (1975)
- Tsytovich, *Theory of Turbulent Plasma* (1977)
- Melrose, *Plasma Astrophysics* (1980)
- Sitenko, *Fluctuations and Nonlinear Wave Interactions in Plasmas* (1982)

1D, unmagnetized, zero net electric field, electrostatic Vlasov-Poisson equation

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} + \frac{e_a E}{m_a}\frac{\partial}{\partial v}\right) f_a = 0,$$
$$\frac{\partial E}{\partial x} = 4\pi\hat{n}\sum_a e_a \int dv\; f_a.$$

Separation into Average and Fluctuation

$$f_a(x,v,t) = F_a(v,t) + \delta f_a(x,v,t),$$
$$E(x,t) = \delta E(x,t).$$

Rewrite the equations

$$\left(\frac{\partial}{\partial t} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right) F_a + \left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\delta f_a = 0,$$

$$\frac{\partial}{\partial x}\,\delta E = 4\pi\hat{n}\sum_a e_a \int dv\;\delta f_a.$$

Fundamentals of Plasma Kinetic Theory

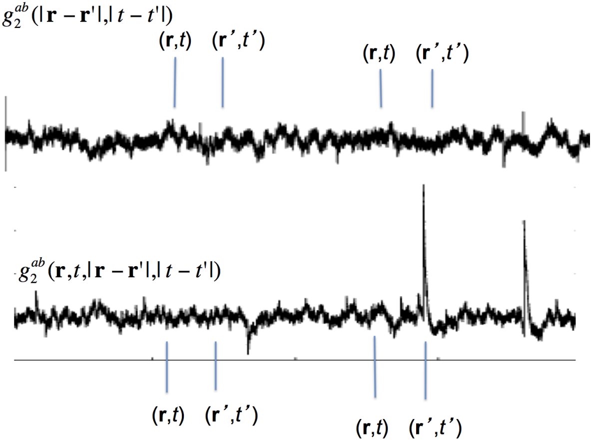
Random phase approximation

$$< \delta f_a >= 0, \qquad < \delta E >= 0.$$

This also implies no coherent nonlinearity.
Stationary and homogeneous turbulence,

$$< \delta f_a(x, v, t)\, \delta f_b(x', v', t') >= g(|x - x'|, |t - t'|, v, v').$$

Fundamentals of Plasma Kinetic Theory



$$\left(\frac{\partial}{\partial t}+\frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)F_a+\left(\frac{\partial}{\partial t}+v\,\frac{\partial}{\partial x}+\frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\delta f_a=0,$$

Upon averaging we have

$$\boxed{\frac{\partial F_a}{\partial t}=-\frac{e_a}{m_a}\frac{\partial}{\partial v}\langle\delta f_a\,\delta E\rangle.}$$

This is the *formal* particle kinetic equation.

Insert the *formal* particle kinetic equation to the original equation

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x}\right)\delta f_a = -\frac{e_a}{m_a}\,\delta E\,\frac{\partial F_a}{\partial v} - \frac{e_a}{m_a}\frac{\partial}{\partial v}\,(\delta f_a\,\delta E - <\delta f_a\,\delta E>)\,.$$

Fourier-Laplace transformation:

$$\downarrow \text{ (slow time)}$$

$$\begin{aligned}
\delta f_a(x,v,t) &= \int dk \int_L d\omega\, \delta f^a_{k\omega}(v,t)\, e^{ikx-i\omega t}, \\
\delta f^a_{k\omega}(v,t) &= \frac{1}{(2\pi)^2} \int dx \int_0^\infty dt\, \delta f_a(x,v,t)\, e^{-ikx+i\omega t}, \\
\delta E(x,t) &= \int dk \int_L d\omega\, \delta E_{k\omega}(t)\, e^{ikx-i\omega t}, \\
\delta E_{k\omega}(t) &= \frac{1}{(2\pi)^2} \int dx \int_0^\infty dt\, \delta E(x,t)\, e^{-ikx+i\omega t},
\end{aligned}$$

where $L = (-\infty + i\sigma, +\infty + i\sigma)$ ($\sigma > 0$ and $\sigma \to 0$).

Fundamentals of Plasma Kinetic Theory

$$\delta E_{k\omega}(t) = -i \sum_a \frac{4\pi \hat{n} e_a}{k} \int dv\, \delta f^a_{k\omega}(v,t),$$

$$\frac{\partial F_a(v,t)}{\partial t} = -\frac{e_a}{m_a} \int dk \int d\omega\, \frac{\partial}{\partial v} < \delta E_{-k,-\omega}(t)\, \delta f^a_{k\omega}(v,t) >,$$

$$\left(\omega - kv + i\,\frac{\partial}{\partial t}\right) \delta f^a_{k\omega}(v,t) = -i\,\frac{e_a}{m_a}\,\delta E_{k\omega}(t)\,\frac{\partial F_a(v,t)}{\partial v}$$
$$- \; i\,\frac{e_a}{m_a} \int dk' \int d\omega'\, \frac{\partial}{\partial v} \left[\delta E_{k'\omega'}(t)\, \delta f^a_{k-k'\,\omega-\omega'}(v,t) \right.$$
$$\left. - < \delta E_{k'\omega'}(t)\, \delta f^a_{k-k',\omega-\omega'}(v,t) > \right].$$

New definition for ω (short-cut trick),

$$\omega \to \omega + i\,\partial/\partial t.$$

Short-hand notations:

$$\begin{aligned} K &= (k,\omega), \qquad E_K = \delta E_{k\omega}, \qquad f_K = \delta f^a_{k\omega}, \\ F &= F_a, \qquad \int dK = \int dk \int d\omega, \\ g_K &= -i\,\frac{e_a}{m_a}\frac{1}{\omega - kv + i0}\frac{\partial}{\partial v}. \end{aligned}$$

Equation for the perturbed particle distribution:

$$f_K = g_K\, F\, E_K + \int dK'\, g_K \left(E_{K'}\, f_{K-K'} - < E_{K'}\, f_{K-K'} >\right).$$

Iteration:

$$f_K = f_K^{(1)} + f_K^{(2)} + \cdots, \qquad (f_K^{(n)} \propto E_K^n).$$

$$\begin{aligned} f_K^{(1)} &= g_K\, F\, E_K, \\ f_K^{(2)} &= \int dK'\, g_K \left(E_{K'}\, f_{K-K'}^{(1)} - < E_{K'}\, f_{K-K'}^{(1)} >\right) \\ &= \int dK'\, g_K\, g_{K-K'}\, F \left(E_{K'}\, E_{K-K'} - < E_{K'}\, E_{K-K'} >\right). \end{aligned}$$

Simplified notations:

$$\sum_{1+2=K} = \int dK_1 \int dK_2 \, \delta(K_1 + K_2 - K),$$

$$E_1 = E_{K_1}, \quad E_2 = E_{K_2}, \qquad g_1 = g_{K_1}, \quad g_2 = g_{K_2}.$$

Iterative solution for f_K

$$f_K = g_K \, F \, E_K + \sum_{1+2=K} g_K \, g_2 \, F \, (E_1 E_2 - < E_1 E_2 >)$$

Symmetrized version

$$f_K = g_K \, F \, E_K + \sum_{1+2=K} \frac{1}{2} \, g_{1+2} \, (g_1 + g_2) \, F \, (E_1 \, E_2 - < E_1 \, E_2 >).$$

Insert f_K to Poisson equation,

$$E_K = -i \sum_a \frac{4\pi \hat{n} e_a}{k} \int dv \; f_K \, .$$

Linear and nonlinear susceptibility response functions,

$$
\begin{aligned}
\epsilon(K) &= 1+\sum_a i\,\frac{4\pi e_a \hat{n}}{k}\int dv\; g_K\, F, \\
\chi_a^{(2)}(1|2) &= \frac{i}{2}\,\frac{4\pi e_a \hat{n}}{k_1+k_2}\int dv\; g_{1+2}\,(g_1+g_2)\,F.
\end{aligned}
$$

$$
\begin{aligned}
\epsilon(K) &= 1+\sum_a \frac{\omega_{pa}^2}{k}\int dv\,\frac{\partial F_a/\partial v}{\omega-kv+i0},\\
\chi_a^{(2)}(1|2) &= -\frac{i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{k_1+k_2}\int dv\,\frac{1}{\omega_1+\omega_2-(k_1+k_2)\,v+i0}\\
&\quad\times\frac{\partial}{\partial v}\left[\left(\frac{1}{\omega_1-k_1v+i0}+\frac{1}{\omega_2-k_2v+i0}\right)\frac{\partial F_a}{\partial v}\right].
\end{aligned}
$$

$$0 = \epsilon(K)\, E_K + \sum_{1+2=K} \chi^{(2)}(1|2)\, (E_1 E_2 - < E_1 E_2 >)\,.$$

Multiply $E_{K'}$ and take ensemble average

$$0 = \epsilon(K) < E_K E_{K'} > + \sum_{1+2=K} \chi^{(2)}(1|2) < E_1 E_2 E_{K'} > .$$

Homogeneous and stationary turbulence,

$$< E(x,t)\,E(x',t') >=< E^2 >_{x-x',t-t'}\ .$$

The spectral representation

$$< E_{k\omega}\,E_{k'\omega'} >= \delta(k+k')\,\delta(\omega+\omega') < E^2 >_{k\omega}\ .$$

$$0 = \epsilon(K) < E^2 >_K + \sum_{1+2=K} \chi^{(2)}(1|2) < E_1 E_2 E_{-K} > .$$

The equation is not closed - to obtain $< E^2 >$ one needs $< E^3 >$, to obtain $< E^3 >$ one needs $< E^4 >$, ...

Three-Body Cumulant and Closure of Hierarchy

If E_K is linear eigenmode,

$$\epsilon(K)\,E_K = 0.$$

then by definition $< E_1 E_2 E_{-K} >= 0$.
But for nonlinear system we write $E_K = E_K^{(0)} + E_K^{(1)}$, where $\epsilon(K)\,E_K^{(0)} = 0$.
Then

$$E_K^{(1)} \approx -\frac{1}{\epsilon(K)} \int dK' \chi^{(2)}(K'|K-K') \left(E_{K'}^{(0)} E_{K-K'}^{(0)} - < E_{K'}^{(0)} E_{K-K'}^{(0)} > \right).$$

Three-body correlation,

$$
\begin{aligned}
&< E_{K'} E_{K-K'} E_{-K} > \\
&\approx \underbrace{< E^{(0)}_{K'} E^{(0)}_{K-K'} E^{(0)}_{-K} >}_{\displaystyle \downarrow \atop \displaystyle 0} + < E^{(1)}_{K'} E^{(0)}_{K-K'} E^{(0)}_{-K} > \\
&+ < E^{(0)}_{K'} E^{(1)}_{K-K'} E^{(0)}_{-K} > + < E^{(0)}_{K'} E^{(0)}_{K-K'} E^{(1)}_{-K} > .
\end{aligned}
$$

$$
\begin{aligned}
< E_{K'} E_{K-K'} E_{-K} >= & -\frac{1}{\epsilon(K')} \int dK'' \chi^{(2)}(K''|K' - K'') \\
& \times \left(\langle E_{K''} E_{K'-K''} E_{K-K'} E_{-K} \rangle - \langle E_{K''} E_{K'-K''} \rangle \langle E_{K-K'} E_{-K} \rangle \right) \\
& - \frac{1}{\epsilon(K-K')} \int dK'' \chi^{(2)}(K''|K - K' - K'') \\
& \times \left(\langle E_{K''} E_{K-K'-K''} E_{K'} E_{-K} \rangle - \langle E_{K''} E_{K-K'-K''} \rangle \langle E_{K'} E_{-K} \rangle \right) \\
& - \frac{1}{\epsilon(-K)} \int dK'' \chi^{(2)}(-K''| -K + K'') \\
& \times \left(\langle E_{K'} E_{K-K'} E_{-K''} E_{-K+K''} \rangle - \langle E_{K'} E_{K-K'} \rangle \langle E_{-K''} E_{-K+K''} \rangle \right).
\end{aligned}
$$

where we dropped the superscript (0). To obtain $< E^3 >$ one needs $< E^4 >$.

The (quasi-normal) closure: for homogeneous and stationary turbulence,

$$
\begin{aligned}
&< E_K E_{K'} E_{K''} E_{K'''} >= \delta(K + K' + K'' + K''') \\
&\times \Big[\delta(K + K') < E^2 >_K < E^2 >_{K''} \\
&+ \delta(K + K'') < E^2 >_K < E^2 >_{K'} \\
&+ \delta(K' + K'') < E^2 >_K < E^2 >_{K'} \\
&+ \cancel{< E^4 >_{K;K+K';K+K'+K''}}\Big].
\end{aligned}
$$

Useful symmetry relations:

$$\chi^{(2)}(-1|-2) = \chi^{(2)*}(1|2),$$
$$\chi^{(2)}(1|2) = \chi^{(2)}(2|1),$$
$$\chi^{(2)}(1|2) = \chi^{(2)}(2|1) = -\chi^{(2)}(1+2|-2).$$

Three-body cumulants finally written as products of two-body cumulants (wave intensities)

$$
\begin{aligned}
< E_{K'} E_{K-K'} E_{-K} >= & \frac{2\chi^{(2)}(K'|K-K')}{\epsilon(K')} < E^2 >_{K-K'} < E^2 >_{K} \\
+ & \frac{2\chi^{(2)}(K'|K-K')}{\epsilon(K-K')} < E^2 >_{K'} < E^2 >_{K} \\
- & \frac{2\chi^{(2)*}(K'|K-K')}{\epsilon^*(K)} < E^2 >_{K'} < E^2 >_{K-K'} \,.
\end{aligned}
$$

Spectral balance equation

$$
\begin{aligned}
0 &= \epsilon(K) < E^2 >_K \\
&+2\int dK' \left(\frac{\{\chi^{(2)}(K'|K-K')\}^2}{\epsilon(K')} < E^2 >_{K-K'} < E^2 >_K \right. \\
&+ \frac{\{\chi^{(2)}(K'|K-K')\}^2}{\epsilon(K-K')} < E^2 >_{K'} < E^2 >_K \\
&\left. - \frac{|\chi^{(2)}(K'|K-K')|^2}{\epsilon^*(K)} < E^2 >_{K'} < E^2 >_{K-K'} \right).
\end{aligned}
$$

Reintroduce the slow time dependence,

$$(\omega - \mathbf{k}\cdot\mathbf{v} + i\partial/\partial t)^{-1}.$$

This leads to

$$\epsilon(k,\omega) < E^2 >_{k\omega} \to \epsilon\left(k, \omega + i\frac{\partial}{\partial t}\right) < E^2 >_{k\omega}$$
$$\to \left(\epsilon(k,\omega) + \frac{i}{2}\frac{\partial\epsilon(k,\omega)}{\partial\omega}\frac{\partial}{\partial t}\right) < E^2 >_{k\omega} .$$

$$
\begin{aligned}
0 \;=\; & \frac{i}{2}\frac{\partial\epsilon(k,\omega)}{\partial\omega}\frac{\partial}{\partial t} < E^2 >_{k\omega} + \epsilon(k,\omega) < E^2 >_{k\omega} \\
& + 2\int dk' \int d\omega' \; \left(\frac{\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2}{\epsilon(k',\omega')} \right. \\
& \times \left\langle E^2 \right\rangle_{k-k',\omega-\omega'} \left\langle E^2 \right\rangle_{k\omega} \\
& + \frac{\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2}{\epsilon(k-k',\omega-\omega')} \left\langle E^2 \right\rangle_{k'\omega'} \left\langle E^2 \right\rangle_{k\omega} \\
& \left. - \frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} \left\langle E^2 \right\rangle_{k'\omega'} \left\langle E^2 \right\rangle_{k-k',\omega-\omega'} \right) .
\end{aligned}
$$

Particle kinetic equation,

$$\frac{\partial F_a}{\partial t} = -\frac{e_a}{m_a} \int dK \, \frac{\partial}{\partial v} < E_{-K} \, f_K^a >,$$

where

$$f_K = f_K^{(1)} + \cancel{f_K^{(2)} + \cdots}.$$

$$\frac{\partial F_a}{\partial t} = \mathrm{Re} \, i \, \frac{e_a^2}{m_a^2} \frac{\partial}{\partial v} \int dk \int d\omega \, \frac{< E^2 >_{k\omega}}{\omega - kv + i0} \frac{\partial F_a}{\partial v}.$$

Summary

Particle Kinetic Equation

$$\frac{\partial F_a}{\partial t} = \mathrm{Re}\, i \frac{e_a^2}{m_a^2} \frac{\partial}{\partial v} \int dk \int d\omega \, \frac{< E^2 >_{k\omega}}{\omega - kv + i0} \frac{\partial F_a}{\partial v}.$$

Dispersion Relation

$$\mathrm{Re}\, \epsilon(k,\omega) < E^2 >_{k\omega} = 0.$$

Wave Kinetic Equation

$$\frac{\partial}{\partial t} < E^2 >_{k\omega} = -\frac{2\,\mathrm{Im}\,\epsilon(k,\omega)}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega} < E^2 >_{k\omega}$$
$$-\frac{4}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega}\,\mathrm{Im}\int dk'\int d\omega'$$
$$\times\left[\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2\left(\frac{< E^2 >_{k-k',\omega-\omega'}}{\epsilon(k',\omega')}\right.\right.$$
$$\left.+\frac{< E^2 >_{k'\omega'}}{\epsilon(k-k',\omega-\omega')}\right) < E^2 >_{k\omega}$$
$$\left.-\frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} < E^2 >_{k'\omega'} < E^2 >_{k-k',\omega-\omega'}\right].$$

Summary

In the next lecture, I will discuss the solution of the plasma weak turbulence theory and demonstrate that the time-asymptotic state, or the "turbulent equilibrium" for the electrons interacting with the Langmuir turbulence in a dynamically steady-state fashion, is none other than the kappa distribution with the kappa index $\kappa = 9/4$.