Kappa Distributions: Turbulent Equilibria

2. Beam-Plasma Instability & Langmuir Turbulence

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Table of Contents

Dispersion Relation

$$\mathrm{Re}\,\epsilon(k,\omega) < E^2 >_{k\omega} = 0.$$

Particle Kinetic Equation

$$\frac{\partial F_a}{\partial t} = \mathrm{Re}\, i\, \frac{e_a^2}{m_a^2} \frac{\partial}{\partial v} \int dk \int d\omega \, \frac{< E^2 >_{k\omega}}{\omega - kv + i0} \frac{\partial F_a}{\partial v}.$$

Wave Kinetic Equation

$$\frac{\partial}{\partial t} < E^2 >_{k\omega} = -\frac{2\,\mathrm{Im}\,\epsilon(k,\omega)}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega} < E^2 >_{k\omega}$$
$$-\frac{4}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega}\,\mathrm{Im}\int dk'\int d\omega'$$
$$\times\left[\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2\left(\frac{< E^2 >_{k-k',\omega-\omega'}}{\epsilon(k',\omega')}\right.\right.$$
$$\left.+\frac{< E^2 >_{k'\omega'}}{\epsilon(k-k',\omega-\omega')}\right) < E^2 >_{k\omega}$$
$$\left.-\frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} < E^2 >_{k'\omega'} < E^2 >_{k-k',\omega-\omega'}\right].$$

Weak Turbulence Theory

$$\mathrm{Re}\,\epsilon(k,\omega) < E^2 >_{k\omega} = 0, \qquad \Rightarrow \qquad \omega = \omega_k^\alpha \quad (\alpha = L, S).$$

$$< E^2 >_{k\omega} = \sum_\alpha \left[I_k^{+\alpha}\,\delta(\omega - \omega_k^\alpha) + I_k^{-\alpha}\,\delta(\omega + \omega_k^\alpha) \right]$$

$$= \sum_{\sigma=\pm1} \sum_\alpha I_k^{\sigma\alpha}\,\delta(\omega - \sigma\omega_k^\alpha).$$

Langmuir ($\alpha = L$) and ion-sound wave ($\alpha = S$)

$$\omega_k^L = \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_{De}^2\right) = \omega_{pe}\left(1 + \frac{3}{4}\frac{k^2 v_{Te}^2}{\omega_{pe}^2}\right).$$

$$\omega_{-k}^L = -\omega_k^L,$$

$$\omega_k^S = \frac{kc_S\,(1 + 3T_i/T_e)^{1/2}}{(1 + k^2\lambda_{De}^2)^{1/2}}, \qquad \omega_{-k}^S = -\omega_k^S.$$

Weak Turbulence Theory

Particle kinetic equation,

$$\frac{\partial F_a}{\partial t} = \frac{\partial}{\partial v}\left(D\,\frac{\partial F_a}{\partial v}\right),$$

$$D = \frac{\pi e_a^2}{m_a^2}\sum_{\sigma=\pm1}\sum_{\alpha}\int dk\; I_k^{\sigma\alpha}\,\delta(\sigma\omega_k^{\alpha} - kv).$$

Weak Turbulence Theory

Wave kinetic equation

$$< E^2 >_{k\omega} = \sum_{\sigma=\pm1} \sum_{\alpha} I_k^{\sigma\alpha}\, \delta(\omega - \sigma\omega_k^{\alpha}),$$

$$< E^2 >_{k'\omega'} = \sum_{\sigma'=\pm1} \sum_{\beta} I_{k'}^{\sigma'\beta}\, \delta(\omega' - \sigma'\omega_{k'}^{\beta}),$$

$$< E^2 >_{k-k',\omega-\omega'} = \sum_{\sigma''=\pm1} \sum_{\gamma} I_{k-k'}^{\sigma''\gamma}\, \delta(\omega - \omega' - \sigma''\omega_{k-k'}^{\gamma}),$$

$$\frac{1}{\epsilon(k,\omega)} = \mathcal{P}\,\frac{1}{\epsilon(k,\omega)} - \sum_{\alpha}\sum_{\sigma=\pm1}\frac{i\pi\,\delta(\omega-\sigma\omega_k^\alpha)}{\partial \mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha},$$

$$\frac{1}{\epsilon^*(k,\omega)} = \mathcal{P}\,\frac{1}{\epsilon^*(k,\omega)} + \sum_{\alpha}\sum_{\sigma=\pm1}\frac{i\pi\,\delta(\omega-\sigma\omega_k^\alpha)}{\partial \mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}.$$

$$\frac{\partial I_k^{\sigma\alpha}}{\partial t} = -\frac{2\,\mathrm{Im}\,\epsilon(k,\sigma\omega_k^\alpha)}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\, I_k^{\sigma\alpha}$$

$$-\frac{4}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\,\mathrm{Im}\int dk'$$

$$\times\Bigg[\sum_{\sigma''=\pm1}\sum_\gamma \mathcal{P}\,\frac{\{\chi^{(2)}(k',\sigma\omega_k^\alpha-\sigma''\omega_{k-k'}^\gamma|k-k',\sigma''\omega_{k-k'}^\gamma)\}^2}{\epsilon(k',\sigma\omega_k^\alpha-\sigma''\omega_{k-k'}^\gamma)}\, I_{k-k'}^{\sigma''\gamma}\, I_k^{\sigma\alpha}$$

$$+\sum_{\sigma'=\pm1}\sum_\beta \mathcal{P}\,\frac{\{\chi^{(2)}(k',\sigma'\omega_{k'}^\beta|k-k',\sigma\omega_k^\alpha-\sigma'\omega_{k'}^\beta)\}^2}{\epsilon(k-k',\sigma\omega_k^\alpha-\sigma'\omega_{k'}^\beta)}\, I_{k'}^{\sigma'\beta}\, I_k^{\sigma\alpha}\Bigg]$$

Weak Turbulence Theory

$$
\begin{aligned}
&+\frac{4\pi}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\sum_{\sigma',\sigma''=\pm1}\sum_{\beta,\gamma} \\
&\times\mathrm{Im}\int dk'\,\Big[\{\chi^{(2)}(k',\sigma'\omega_{k'}^\beta|k-k',\sigma''\omega_{k-k'}^\gamma)\}^2 \\
&\times\left(\frac{I_{k-k'}^{\sigma''\gamma}\,I_k^{\sigma\alpha}}{\partial\,\mathrm{Re}\,\epsilon(k',\sigma'\omega_{k'}^\beta)/\partial\sigma'\omega_{k'}^\beta}+\frac{I_{k'}^{\sigma'\beta}\,I_k^{\sigma\alpha}}{\partial\,\mathrm{Re}\,\epsilon(k-k',\sigma''\omega_{k-k'}^\gamma)/\partial\sigma''\omega_{k-k'}^\gamma}\right) \\
&+\frac{|\chi^{(2)}(k',\sigma'\omega_{k'}^\beta|k-k',\sigma''\omega_{k-k'}^\gamma)|^2}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\,I_{k'}^{\sigma'\beta}I_{k-k'}^{\sigma''\gamma}\Big] \\
&\times\delta(\sigma\omega_k^\alpha-\sigma'\omega_{k'}^\beta-\sigma''\omega_{k-k'}^\gamma).
\end{aligned}
$$

Weak Turbulence Theory

Evaluations of nonlinear susceptibilites must be done. See the following references:

- P. H. Yoon, Generalized weak turbulence theory, POP, 7, 4858 (2000)
- L. F. Ziebell, R. Gaelzer, and P. H. Yoon, Nonlinear development of weak beam-plasma instability, POP, 8, 3982 (2001)
- P. H. Yoon, Effects of spontaneous fluctuations on the generalized weak turbulence theory, POP, 12, 042306 (2005)
- P. H. Yoon, T. Rhee, and C.-M. Ryu, Effects of spontaneous thermal fluctuations on nonlinear beam-plasma interaction, POP, 12, 062310 (2005)

Weak Turbulence Theory

Induced Emission: Linear Wave-Particle Resonance

$$
\begin{aligned}
\left.\frac{\partial I_k^{\sigma L}}{\partial t}\right|_{\text{ind. emiss.}} &= \pi\,\sigma\omega_k^L\,\frac{\omega_{pe}^2}{k}\int dv\;\delta(\sigma\omega_k^L - kv)\,\frac{\partial F_e}{\partial v}\,I_k^{\sigma L},\\
\left.\frac{\partial}{\partial t}\right|_{\text{ind. emiss.}}\frac{I_k^{\sigma S}}{\mu_k} &= \pi\,\mu_k\,\sigma\omega_k^L\,\frac{\omega_{pe}^2}{k}\int dv\;\delta(\sigma\omega_k^S - kv)\,\frac{\partial}{\partial v}\\
&\quad\times\left(F_e + \frac{m_e}{m_i}\,F_i\right)\frac{I_k^{\sigma S}}{\mu_k},\\
\mu_{\mathbf{k}} &= k^3\lambda_{De}^3\sqrt{\frac{m_e}{m_i}}\sqrt{1+\frac{3T_i}{T_e}}.
\end{aligned}
$$

Decay/Coalescence: Nonlinear Three-Wave Resonance

$$\frac{\partial I_k^{\sigma L}}{\partial t}\bigg|_{\rm decay} = \frac{\pi}{2}\frac{e^2}{T_e^2}\,\sigma\omega_k^L \sum_{\sigma',\sigma''=\pm1}\int dk'\,\frac{\mu_{k-k'}}{(k-k')^2} \times\left[\sigma\omega_k^L\,I_{k'}^{\sigma' L}\,\frac{I_{k-k'}^{\sigma'' S}}{\mu_{k-k'}} - \left(\sigma'\omega_{k'}^L\,\frac{I_{k-k'}^{\sigma'' S}}{\mu_{k-k'}} + \sigma''\omega_{k-k'}^L\,I_{k'}^{\sigma' L}\right) I_k^{\sigma L}\right] \times\delta(\sigma\omega_k^L - \sigma'\omega_{k'}^L - \sigma''\omega_{k-k'}^S),$$

Weak Turbulence Theory

$$\frac{\partial}{\partial t}\bigg|_{\rm decay}\frac{I_k^{\sigma S}}{\mu_k}=\frac{\pi}{4}\frac{e^2}{T_e^2}\,\sigma\omega_k^S\sum_{\sigma',\sigma''=\pm1}\int dk'\,\frac{\mu_k}{k^2}$$
$$\times\left[\sigma\omega_k^L\,I_{k'}^{\sigma' L}\,I_{k-k'}^{\sigma'' L}-\left(\sigma'\omega_{k'}^L\,I_{k-k'}^{\sigma'' L}+\sigma''\omega_{k-k'}^L\,I_{k'}^{\sigma' L}\right)\frac{I_k^{\sigma S}}{\mu_k}\right]$$
$$\times\delta(\sigma\omega_k^S-\sigma'\omega_{k'}^L-\sigma''\omega_{k-k'}^L).$$

Induced Scattering: Nonlinear Wave-Particle Resonance

$$\left.\frac{\partial I_k^{\sigma L}}{\partial t}\right|_{\rm scatt.} = \sigma\omega_k^L \frac{\pi}{\omega_{pe}^2} \frac{e^2}{m_e\, m_i} \sum_{\sigma'=\pm 1} \int dk' \int dv$$
$$\times\, (k-k')\, \frac{\partial F_i}{\partial v}\, \delta[\sigma\omega_k^L - \sigma'\omega_{k'}^L - (k-k')\, v]\, I_{k'}^{\sigma' L}\, I_k^{\sigma L}.$$

Adding effects of spontaneous thermal fluctuation

This tutorial is too short to discuss in detail but in general, induced processes must be balanced by spontaneous processes.

$$\begin{aligned}
\frac{\partial F_e}{\partial t} &= \frac{\partial}{\partial v_i}\left(A_i\,F_e + D_{ij}\,\frac{\partial F_e}{\partial v_j}\right), \\
A_i &= \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}), \\
D_{ij} &= \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i\,k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\,I_{\mathbf{k}}^{\sigma L}.
\end{aligned}$$

Langmuir Wave Kinetic Equation

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\underbrace{\frac{\hat{n}\,e^2}{\pi}\,F_e}_{\text{spont. emission}} + \underbrace{\sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}}^{\sigma L}\,\mathbf{k}\cdot\frac{\partial F_e}{\partial \mathbf{v}}}_{\text{induced emission}}\right)$$

$$+2\sum_{\sigma',\sigma''=\pm 1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'\,\frac{\pi}{4}\,\frac{e^2}{T_e^2}\,\frac{\mu_{\mathbf{k}-\mathbf{k}'}\,(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2\,|\mathbf{k}-\mathbf{k}'|^2}$$

$$\times\,\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times\left(\underbrace{\sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}_{\text{spont. decay}}\underbrace{-\sigma'\omega_{\mathbf{k}'}^L\,I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}\,I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\,I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}}^{\sigma L}}_{\text{induced decay}}\right)$$

Weak Turbulence Theory

$$
-\frac{\pi e^2}{m_e^2\,\omega_{pe}^2}\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2}\,\delta[\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]
$$

$$
\times\Bigg[\underbrace{\frac{\hat{n}\,e^2}{\pi\omega_{pe}^2}\,\sigma\omega_{\mathbf{k}}^L\left(\sigma'\omega_{\mathbf{k}'}^L\,I_{\mathbf{k}}^{\sigma L}-\sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}'}^{\sigma' L}\right)(F_e+F_i)}_{\text{spontaneous scattering}}
$$

$$
+\underbrace{I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}}^{\sigma L}\,(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial}{\partial\mathbf{v}}\left(\left(\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L\right)F_e-\frac{m_e}{m_i}\left(\sigma\omega_{\mathbf{k}}^L\right)F_i\right)}_{\text{induced scattering}}\Bigg]\,.
$$

Forward/backward-Ion-sound Wave Kinetic Equation

$$\frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t} = \frac{\pi \mu_{\mathbf{k}} \omega_{pe}^2}{k^2} \int d\mathbf{v}\, \delta(\sigma\omega_{\mathbf{k}}^S - \mathbf{k}\cdot\mathbf{v})$$

$$\times \Bigg[\underbrace{\frac{\hat{n}\, e^2}{\pi} (F_e + F_i)}_{\text{spont. emission}} + \underbrace{\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}}^{\sigma S} \left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left(F_e + \frac{m_e}{m_i} F_i\right)}_{\text{induced emission}} \Bigg]$$

$$\sum_{\sigma',\sigma''=\pm 1} \sigma\omega_{\mathbf{k}}^L \int d\mathbf{k}'\, \frac{\pi}{4} \frac{e^2}{T_e^2} \frac{\mu_{\mathbf{k}}\, [\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2\, k'^2\, |\mathbf{k}-\mathbf{k}'|^2}\, \delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L)$$

$$\times \Big(\underbrace{\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}}_{\text{spont. decay}} \underbrace{- \sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}\, I_{\mathbf{k}}^{\sigma S} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}}^{\sigma S}}_{\text{induced decay}} \Big).$$

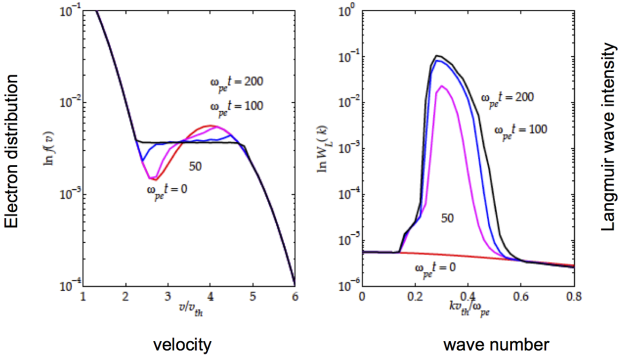
Quasi-stationary ions

$$F_i = \frac{e^{-v^2/v_{Ti}^2}}{\pi^{1/2}\, v_{Ti}}.$$

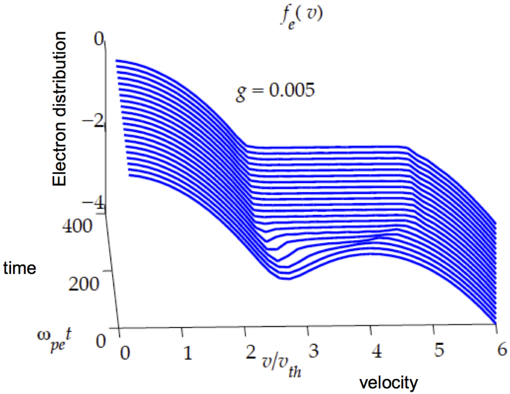
Initial beam plus background electron distribution

$$F_e(v,0) = \frac{(1-\delta)\, e^{-v^2/v_{Te}^2}}{\pi^{1/2}\, v_{Te}} + \frac{\delta\, e^{-(v-v_0)^2/v_{Te}^2}}{\pi^{1/2}\, v_{Te}}.$$

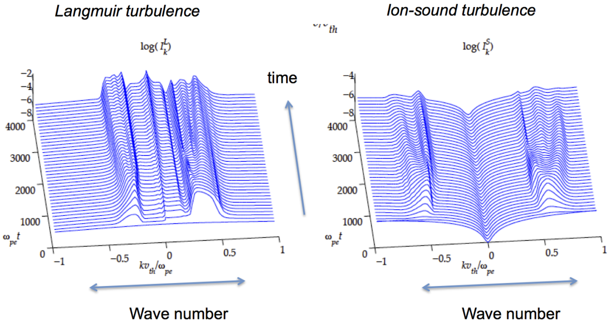
Beam-Plasma Instability: Quasi-Linear Saturation Regime



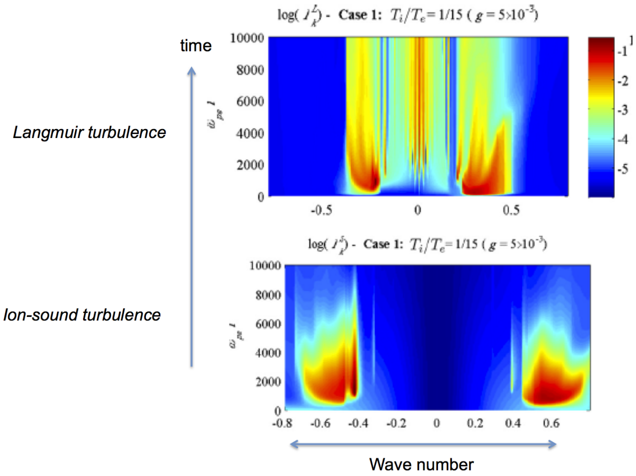
Beam-Plasma Instability: Nonlinear Mode-Coupling Regime



Beam-Plasma Instability: Nonlinear Mode-Coupling Regime

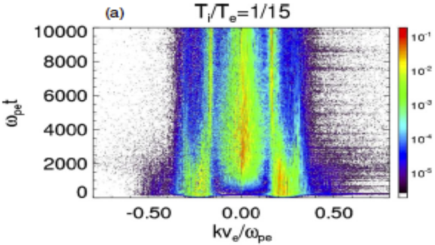


Beam-Plasma Instability and Langmuir Turbulence

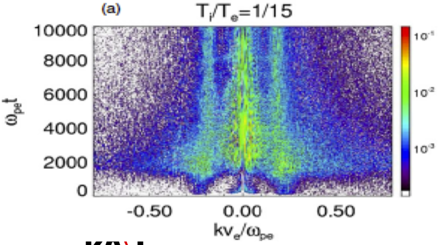


Beam-Plasma Instability and Langmuir Turbulence

Langmuir turbulence

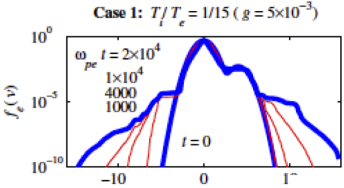


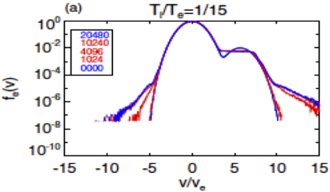
Ion-sound turbulence



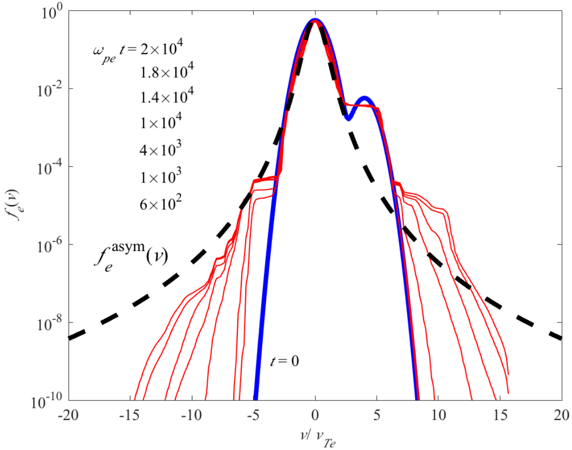
Beam-Plasma Instability and Langmuir Turbulence

Formation of energetic tail in the near the end of nonlinear regime (kappa VDF?)





Beam-Plasma Instability and Langmuir Turbulence



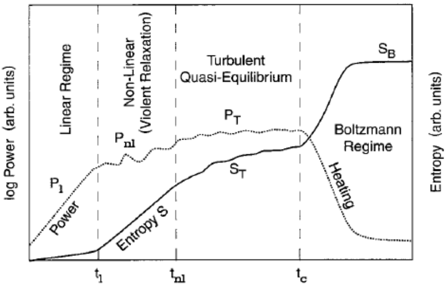
Beam-Plasma Instability and Langmuir Turbulence

Physica Scripta. Vol. 59, 19–26, 1999

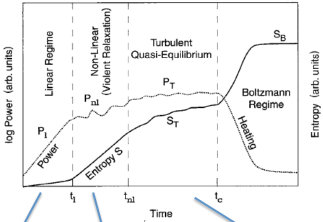
Kinetic Theoretical Foundation of Lorentzian Statistical Mechanics

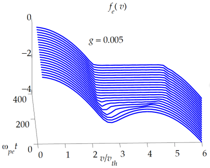
Rudolf A. Treumann*

FIGURES

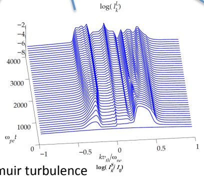


Beam-Plasma Instability and Langmuir Turbulence

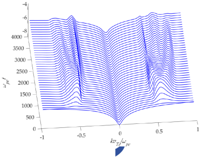




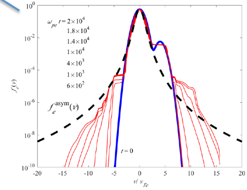
Beam relaxation



Langmuir turbulence



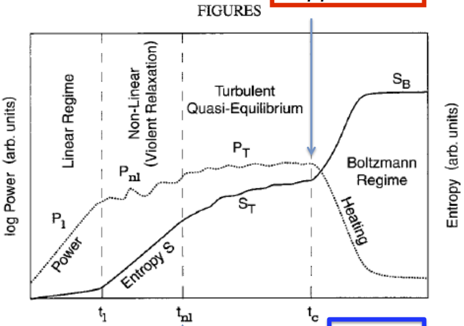
Ion sound turbulence



Suprathermal tail Generation and kappa distribution

Turbulent Quasi-Equilibrium

Kappa VDF?



What is t_c?

$t_1 \sim 100$ - $200\ \omega_{pe}^{-1}$

$t_{nl} \sim 10^4\ \omega_{pe}^{-1}$

Turbulent Equilibrium and Kappa Distribution

Asymptotic solution ($\partial/\partial t = 0$)

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_i}\left(A_i f_e + D_{ij}\frac{\partial f_e}{\partial v_j}\right),$$

$$A_i = \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}),$$

$$D_{ij} = \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})I_{\mathbf{k}}^{\sigma L}.$$

Linear wave-particle

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right)$$

Wave-wave

$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}' V_{\mathbf{k},\mathbf{k}'}^L\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$
$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$

Nonlinear wave-particle

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$
$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})f_i - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L}I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right)$$

Steady-State Solution (Quasi-Equilibrium)

Electron kinetic equation

$$0 = \frac{\partial}{\partial v_i}\left(v_i G f_e + D \frac{v_i v_j}{v^2} \frac{\partial f_e}{\partial v_j} \right),$$

$$G = \frac{e^2 \omega_{pe}^2}{4\pi m_e v^2} \int \frac{d\mathbf{k}}{k^2} \delta(\omega_{pe} - \mathbf{k}\cdot \mathbf{v}),$$

$$D = \frac{\pi e^2 \omega_{pe}^2}{m_e^2 v^2} \int \frac{d\mathbf{k}}{k^2} \delta(\omega_{pe} - \mathbf{k}\cdot \mathbf{v}) I(\mathbf{k}).$$

Steady-state solution [Gurevich, 1960; Hasegawa et al., 1985]

$$f_e = C \exp\left(- \int dv \frac{vG}{D} \right).$$

Steady-State Wave Equation

$$0 = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\omega_k - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \omega_{pe}I(\mathbf{k})\mathbf{k}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\right)$$

$$-\frac{\omega_{pe}}{4\pi n T_i}\int d\mathbf{k}'\int d\mathbf{v}\,\delta[\omega_k - \omega_{k'} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

$$\times\left(\frac{T_i}{4\pi^2}[\omega_{k'}I(\mathbf{k}) - \omega_k I(\mathbf{k}')] + I(\mathbf{k})I(\mathbf{k}')(\omega_k - \omega_{k'})\right)f_i.$$

Balance of spontaneous and induced *emissions*

$$0 \;\; \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\omega_k - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \omega_{pe} I(\mathbf{k})\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right) = 0$$

$$-\frac{\omega_{pe}}{4\pi n T_i}\int d\mathbf{k}'\int d\mathbf{v}\,\delta[\omega_k-\omega_{k'}-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$
$$\times\left(\frac{T_i}{4\pi^2}[\omega_{k'}I(\mathbf{k})-\omega_k I(\mathbf{k}')] + I(\mathbf{k})I(\mathbf{k}')(\omega_k-\omega_{k'})\right)f_i. \qquad = 0$$

Balance of spontaneous and induced *scatterings*

Turbulent Equilibrium and Kappa Distribution

Balance of particle equation and linear wave equation leads to

$$f_e(v) \propto \left[1+\frac{v^2}{\left(\kappa-\frac{3}{2}\right)v_{Te}^2}\right]^{-\kappa-1},$$

$$I(k) = \frac{T_e}{4\pi^2}\frac{\kappa-\frac{3}{2}}{\kappa+1}\left(1+\frac{\omega_{pe}^2}{\left(\kappa-\frac{3}{2}\right)(kv_{Te})^2}\right).$$

Balance of nonlinear wave equation separately leads to

$$I(k) = \frac{T_i}{4\pi^2}\left(1+\frac{4}{3}\frac{\omega_{pe}^2}{k^2v_{Te}^2}\right).$$

Turbulent Equilibrium and Kappa Distribution

The two alternative expressions for the wave intensity,

$$I(k) = \frac{T_e}{4\pi^2}\frac{\kappa-\frac{3}{2}}{\kappa+1}\left(1+\frac{\omega_{pe}^2}{\left(\kappa-\frac{3}{2}\right)(kv_{Te})^2}\right) \qquad \text{and}$$

$$I(k) = \frac{T_i}{4\pi^2}\left(1+\frac{4}{3}\frac{\omega_{pe}^2}{k^2 v_{Te}^2}\right),$$

must be reconciled, which is possible for

$$\boxed{T_e\frac{\kappa-3/2}{\kappa+1} = T_i, \qquad \text{and} \qquad \kappa-\frac{3}{2}=\frac{3}{4},}$$

or equivalently,

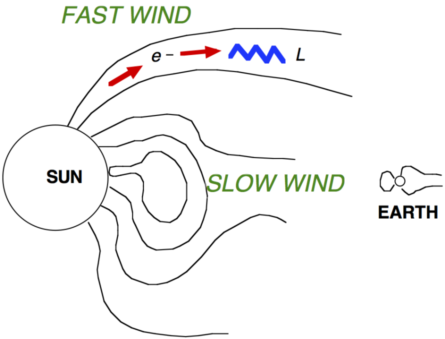
$$\kappa = \frac{9}{4} = 2.25, \qquad \text{or} \qquad q = \frac{\kappa - 1}{\kappa} = \frac{5}{9}.$$

To test this finding, we turn to the solar wind.

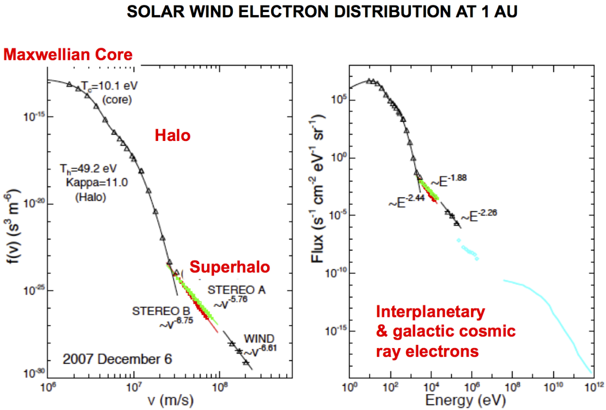
Turbulent Equilibrium and Kappa Distribution

- P. H. Yoon, T. Rhee & C.-M. Ryu, Self-consistent generation of superthermal electrons by beam-plasma interaction, PRL, 95, 215003 (2005)
- P. H. Yoon, Electron kappa distribution and steady-state Langmuir turbulence, POP, 19, 052301 (2012)
- P. H. Yoon, Electron kappa distribution and quasi-thermal noise, JGR, 119, 7074 (2014)

Quiet Time Solar Wind Electrons



Quiet Time Solar Wind Electron Distribution

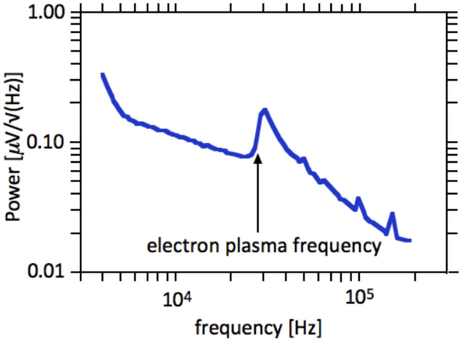


Wang et al., *ApJ Lett.* (2012)

Quiet Time Solar Wind Electron Distribution

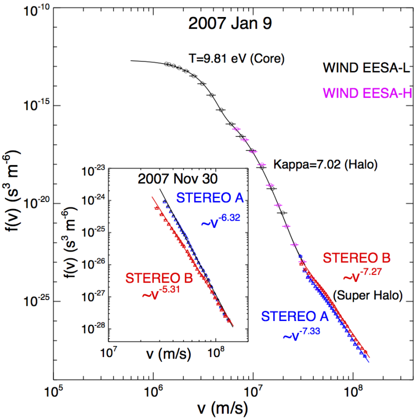
$$I(k)$$

Langmuir
Turbulence
Spectrum



Quasi-thermal noise [by Stuart Bale]

Quiet Time Solar Wind Electron Distribution



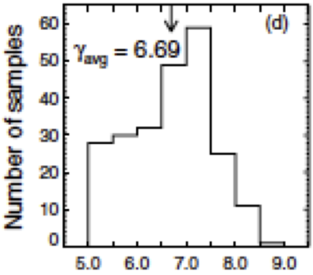
- **Theory:**

$$f_e(v) \approx \frac{1}{v^{2\kappa}} \sim v^{-6.5}.$$

- **Observation:**

$$f_e(v) \approx v^{-5.5} - v^{-7.5}$$

Quiet Time Solar Wind Electron Distribution

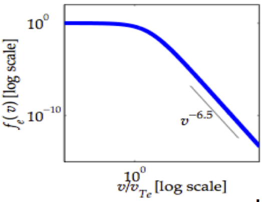


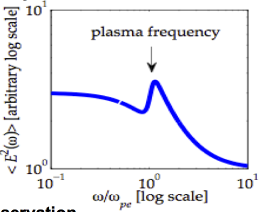
$$f_e(v) \approx v^{-\gamma}$$

- Theory: $f_e(v) \approx v^{-6.5}$

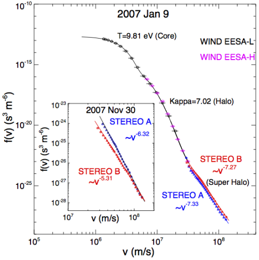
Quiet Time Solar Wind Electron Distribution

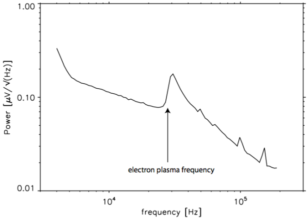
theory





observation





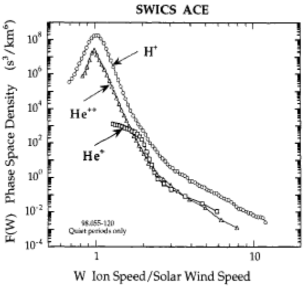
Conclusion

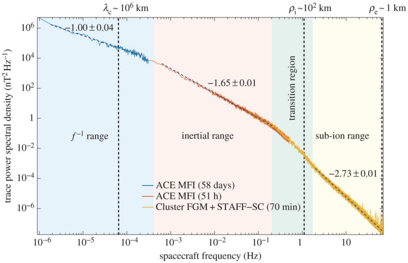
- Kappa distributions may imply non-extensive statistical principles dictating the behavior of complex systems, and systems interacting with long-range force such as plasmas.
- Turbulent equilibrium describes a system of plasma particles constantly exchanging momentum and energy with turbulence, while maintaining a dynamical steady-state. Such a state happens to correspond to kappa distribution.
- Conclusion: turbulent equilibrium state for plasmas may be equivalent to the non-extensive equilibrium.

Conclusion

- The quiet-time solar wind also features inverse power-law proton distribution $f(v) \propto v^{-\alpha}$, where $\alpha \sim 5$ [Gloeckler & Fisk, Fisk & Gloeckler, . . .].
- There is a permanent low-frequency solar wind turbulence of the kinetic Alfvénic variety.
- Question: Are the two dynamically coupled? In other words, are the solar wind protons and kinetic Alfvénic turbulence in dynamical equilibrium?

Conclusion


SWICS ACE
F(W) Phase Space Density (s^3/km^6)
H+
He++
He+
98.055-120
Quiet periods only
W Ion Speed/Solar Wind Speed


λc ~ 10^6 km
ρi ~ 10^2 km
ρe ~ 1 km
−1.00 ± 0.04
−1.65 ± 0.01
−2.73 ± 0.01
transition region
f^−1 range
inertial range
sub-ion range
ACE MFI (58 days)
ACE MFI (51 h)
Cluster FGM + STAFF–SC (70 min)
trace power spectral density (nT^2 Hz^−1)
spacecraft frequency (Hz)