Global Astrometry in the Relativistic Context

History, principles and modeling overview

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INAF - Astrophysical Observatory of Torino

ISSS Course on Space Astromery for Astrophysics
L'Aquila, June 3-7, 2019

Outline

'Abouts' and 'Not abouts'

This talk is NOT about **General Relativity!**

- It doesn't deal with GR principles
- It doesn't show GR typical physics

This talk is NOT about **Astrometry!**

- It doesn't explain astrometric techniques
- It doesn't speak about data reduction

This talk IS about **Models** and (just barely) **Numerical techniques!**

- It will introduce the subject of Global and Absolute Astrometry from the "abstract" mathematical point of view
- It will give an (historical) overview of the different solutions conceived so far
- It will show the principles that must applied to embed GR into the modeling equations
- It will be "Gaia-centric"!

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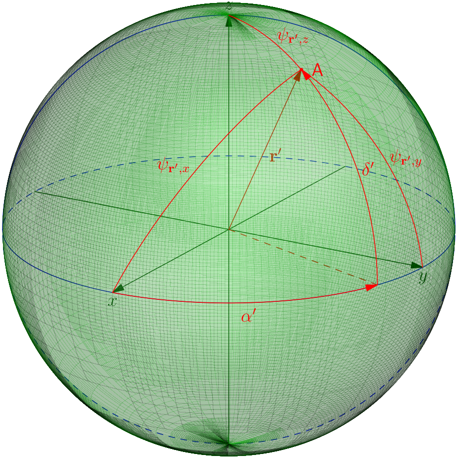
What is astrometry about

- Measure the "apparent" (i.e. local) position (α', δ') or $\psi_{\mathbf{r}',\mathbf{e}_a}$ of a star.
- Get its "true" (i.e. barycentric) position (α, δ) from them. This requires the application of some "corrections".

Mathematical modeling

Write the equations that connect the observables (i.e. $\cos\psi_{\mathbf{r}',\mathbf{e}_a}$) with the unknowns (i.e. α and δ)



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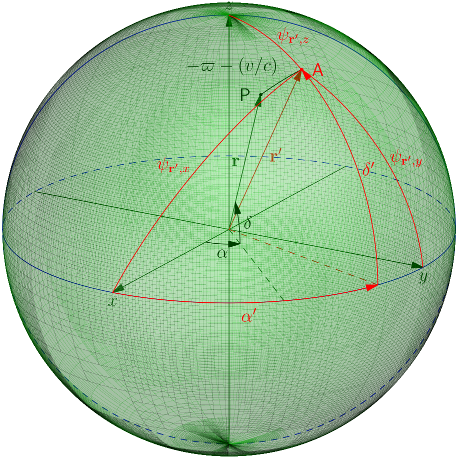

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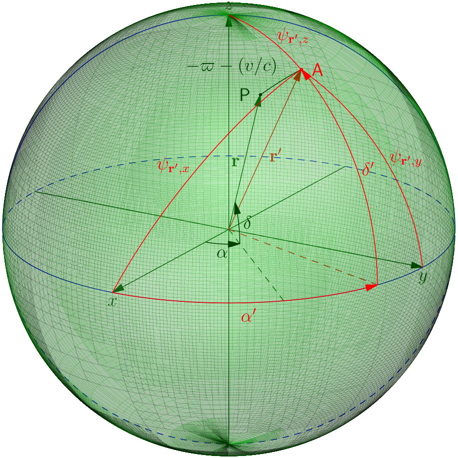
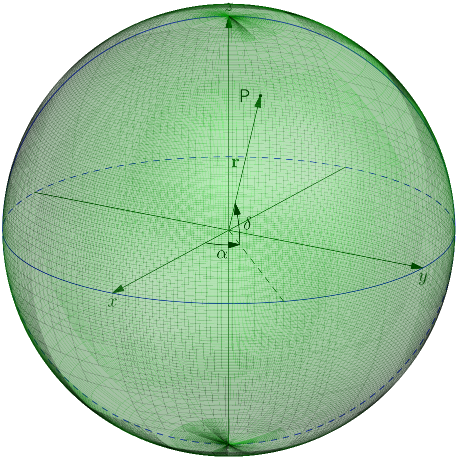


Outline

Reference systems: Geocentric vs. Heliocentric models

1. Geocentric Astrometry
2. Heliocentric theory
3. Search for parallax (annual)
4. They found the aberration first!



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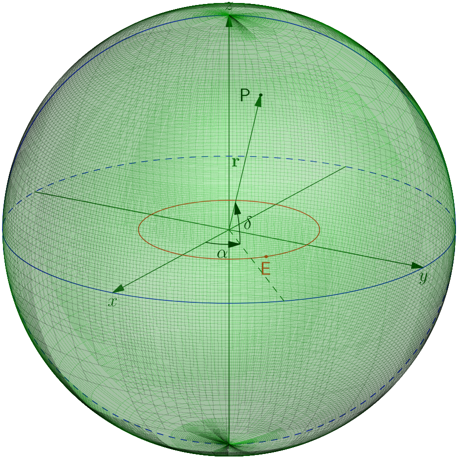

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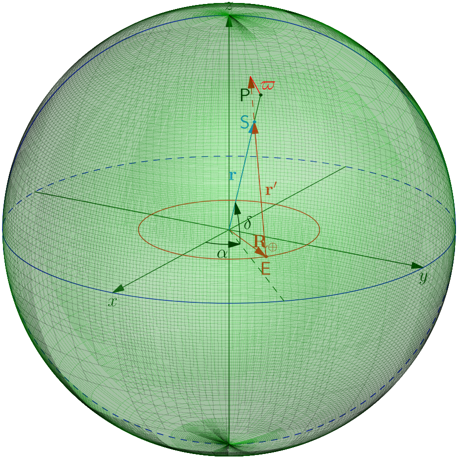

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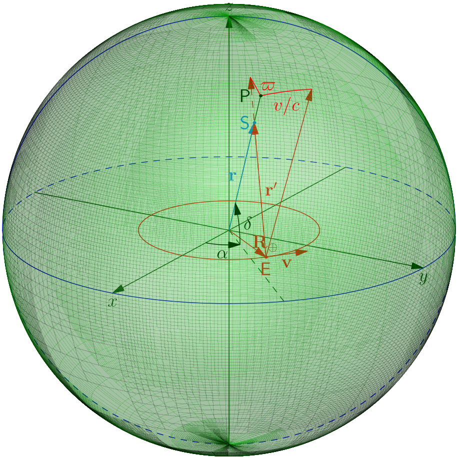


Modeling the observable

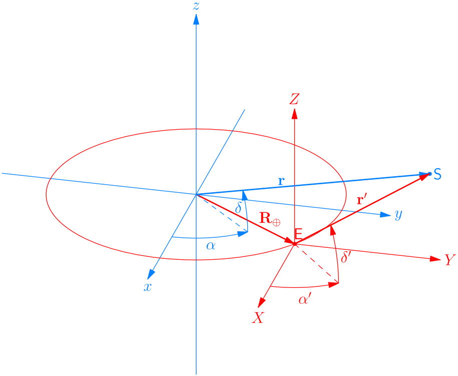
$$\mathbf{a}\cdot\mathbf{b} = ab\cos\psi_{\mathbf{a},\mathbf{b}} \qquad \mathbf{r}' = \mathbf{r} - \mathbf{R}_\oplus$$

Observable in the local reference system:

$$\begin{aligned}
\cos\alpha'\cos\delta' &= \frac{\mathbf{r}'_X}{r'} = \frac{\mathbf{r}'\cdot\hat{\mathbf{X}}}{r'} \equiv \cos\psi_{\mathbf{r}',\hat{\mathbf{X}}} \\
\sin\alpha'\cos\delta' &= \frac{\mathbf{r}'_Y}{r'} = \frac{\mathbf{r}'\cdot\hat{\mathbf{Y}}}{r'} \equiv \cos\psi_{\mathbf{r}',\hat{\mathbf{Y}}} \\
\sin\delta' &= \frac{\mathbf{r}'_Z}{r'} = \frac{\mathbf{r}'\cdot\hat{\mathbf{Z}}}{r'} \equiv \cos\psi_{\mathbf{r}',\hat{\mathbf{Z}}}
\end{aligned}$$

Expressed as a function of the barycentric unknowns ($\varpi = R_\oplus / r$):

$$\begin{aligned}
\mathbf{r}' &= \mathbf{r} - \mathbf{R}_\oplus = r\left(\hat{\mathbf{r}} - \frac{R_\oplus}{r}\hat{\mathbf{R}}_\oplus\right) \\
&= \frac{R_\oplus}{\varpi}\left(\hat{\mathbf{r}}(\alpha,\delta) - \varpi\hat{\mathbf{R}}_\oplus(\lambda_\oplus)\right) \qquad (1)
\end{aligned}$$



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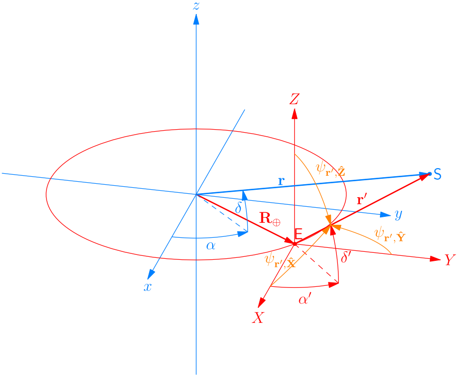

Reference systems: Astrometry and the laws of dynamics

From a dynamical point of view the orbital motion of the Earth opens another issue:

$$\mathbf{v} = \mathbf{v}(t)$$
$$\Downarrow$$
$$\mathbf{a} \neq 0$$

1. Newtonian dynamics: existence of a privileged class of reference systems
2. Astrometry as an experimental tool for seeking an inertial reference system

Astrometry & fundamental physics

The determination of a global inertial reference frame is a problem of **fundamental physics**

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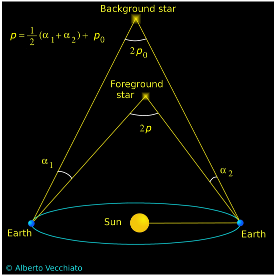
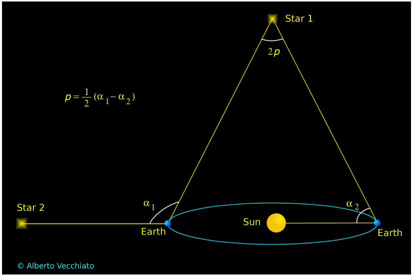
Outline

Absolute vs. relative Astrometry

Relative Astrometry: small field



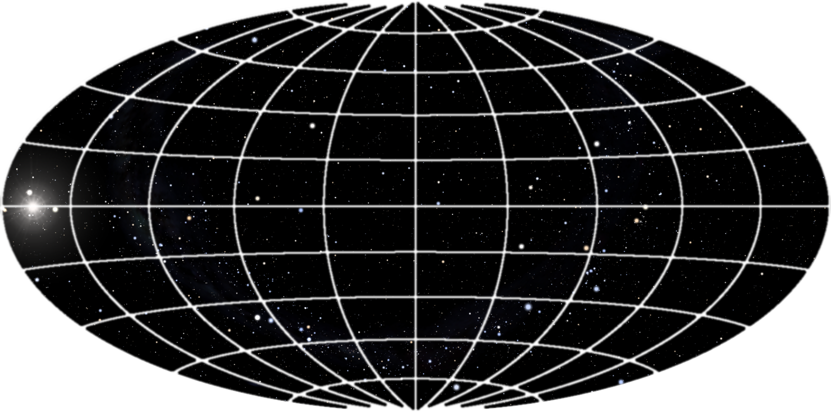
Absolute Astrometry: large angles



HARD to be verified! Absolute catalog≡definition of a unit of measure
⇒calls for verification by a similar measurement campaign

The sky

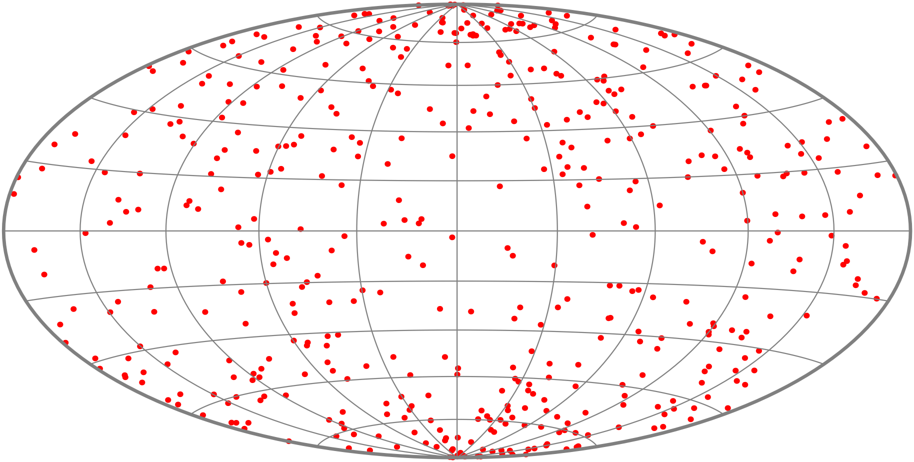
Global reference frame



Outline

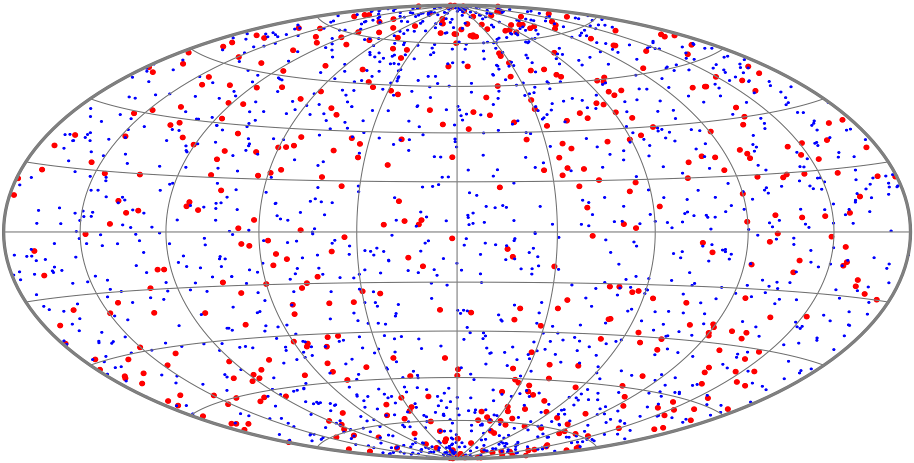
Primaries

The Global Astrometric Sphere is first reconstructed with respect to a subset ($\sim 10^8$ out of $\sim 10^9$) of well-behaved stars called primaries.



Primaries and Secondaries

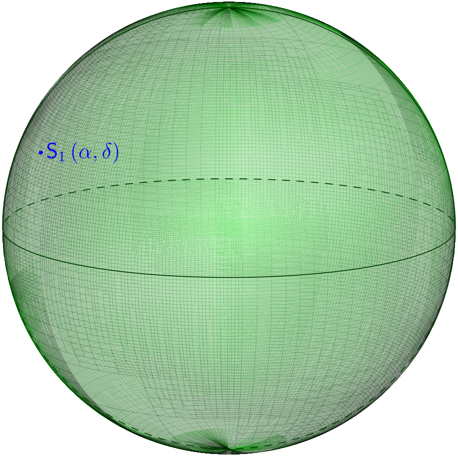
The reference frame materialized by the primaries is used by other pipeline processes to include the other stars into the Gaia sphere.



Outline

Principles of the sphere reconstruction

The ideal picture



Create a "geodetic" network of measurements

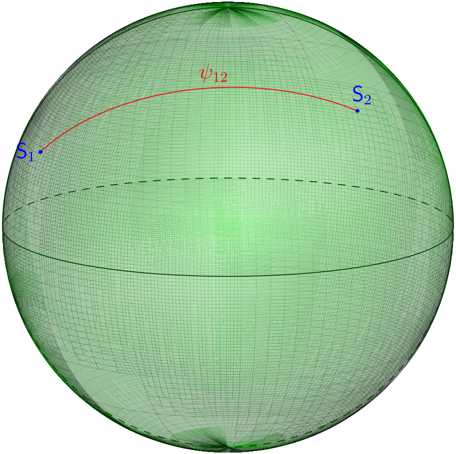
$$N_* = 1$$

$$N_{\rm unk} = 2$$

$$N_{\rm arcs} = 0$$

Principles of the sphere reconstruction

The ideal picture



Create a "geodetic" network of measurements

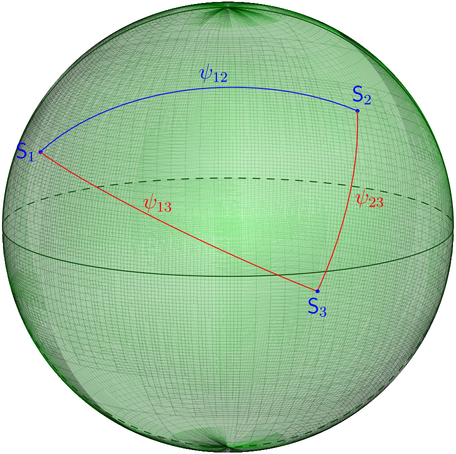
$$N_* = 2$$

$$N_{\text{unk}} = 4$$

$$N_{\text{arcs}} = 1$$

Principles of the sphere reconstruction

The ideal picture



Create a "geodetic" network of measurements

$$N_* = 3$$

$$N_{\rm unk} = 6$$

$$N_{\rm arcs} = 3$$

Principles of the sphere reconstruction

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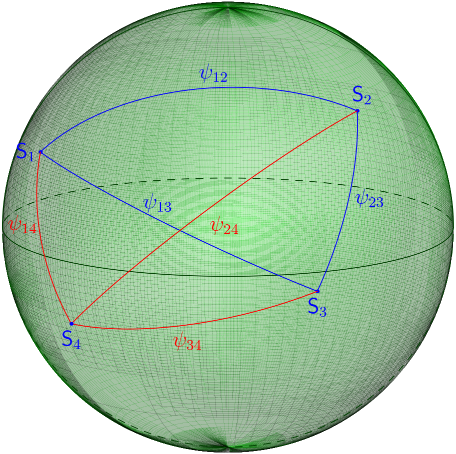


Create a "geodetic" network of measurements

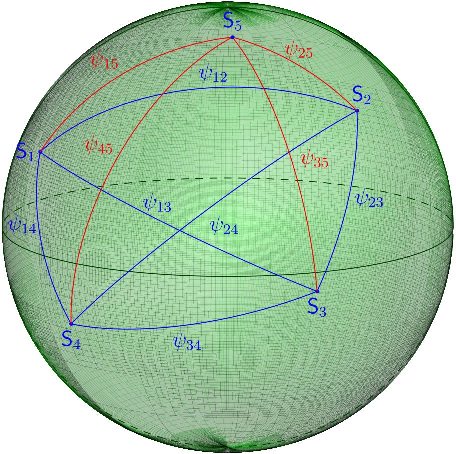
$$N_* = 4$$

$$N_{\rm unk} = 8$$

$$N_{\rm arcs} = 6$$

Principles of the sphere reconstruction

The ideal picture



Create a "geodetic" network of measurements

$$N_* = 5$$

$$N_{\rm unk} = 10$$

$$N_{\rm arcs} = 10$$

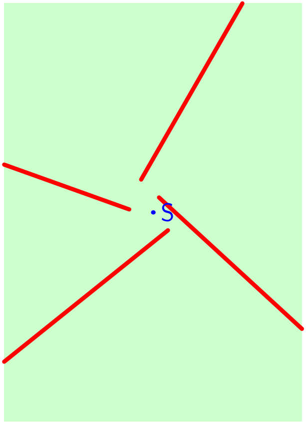
Network closed!
Solve an Equation System

Principles of the sphere reconstruction

The (almost) real picture

Observational errors ⇒

1. solution in the least-squares sense;
2. overdetermined system of equations.



Principles of the sphere reconstruction

The (almost) real picture

Observational errors $\Rightarrow$

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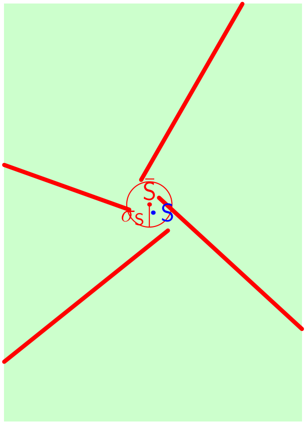


Principles of the sphere reconstruction

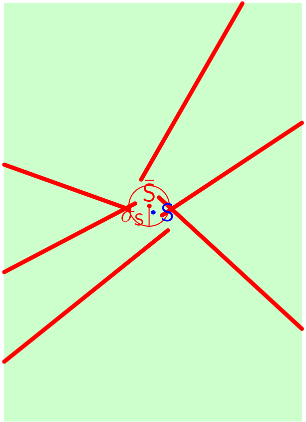
The (almost) real picture

Observational errors $\Rightarrow$

1. solution in the least-squares sense;
2. overdetermined system of equations.

$$N_{\rm unk} \sim N_* \simeq 10^8$$

$$N_{\rm obs} \sim 10^2 N_{\rm unk} \sim 10^{10}$$

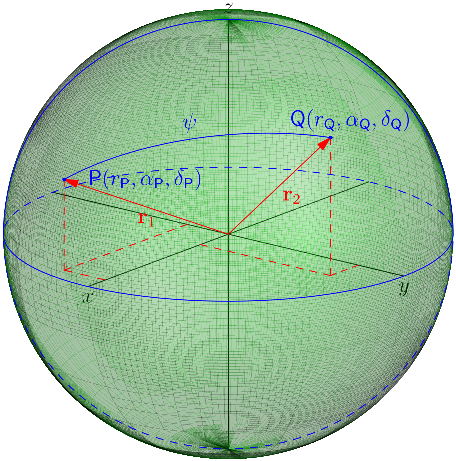


Mathematical modeling: the Euclidean arc

- The basic astrometric observable is an angle between two stars' directions
$$\cos\psi = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|} \qquad (2)$$

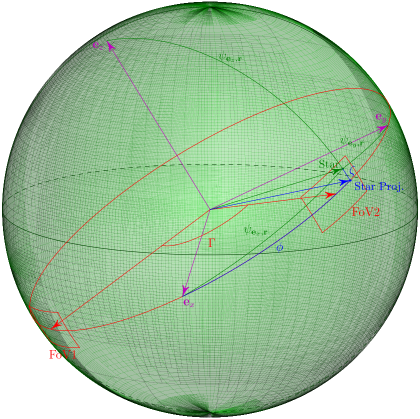
- It depends on the astrometric coordinates of the two stars (Eq. (1)):
$$\begin{aligned} \mathrm{r} &= \mathrm{r}(\alpha, \delta, \varpi) \\ &= \mathrm{r}(\alpha_0, \delta_0, \varpi, \mu_\alpha, \mu_\delta) \end{aligned}$$

- Stellar aberration enters in the definition of the satellite reference system



Outline

Mathematical modeling: the Euclidean abscissa



- The Gaia basic observable is the abscissa ϕ between the x axis and one viewing direction

$$\cos \psi_{(\hat{a},\mathbf{r})} = \frac{\mathbf{e}_{\hat{a}} \cdot \mathbf{r}}{|\mathbf{r}|} \qquad (3)$$

$$\cos \phi = \frac{\cos \psi_{(\hat{x},\mathbf{r})}}{\sqrt{1 - \cos^2 \psi_{(\hat{z},\mathbf{r})}}} \qquad (4)$$

- Depends on the coordinates of one star (S) and on the satellite attitude (A) at the time of the observation
- The aberration enters in the same way as for the arcs

The Linearized system of equations (I)

- The basic equations are highly non-linear

$$\cos\phi = \frac{\cos\psi_{(\hat{x},\mathbf{r})}}{\sqrt{1-\cos^2\psi_{(\hat{z},\mathbf{r})}}} = F\left(\boldsymbol{x}^{\mathrm{S}}, \boldsymbol{x}^{\mathrm{A}}, \boldsymbol{x}^{\mathrm{C}}, \boldsymbol{x}^{\mathrm{G}}\right)$$

- The Equation system can be quite large (up to $\sim 10^{10} \times 10^{8}$)
- Solving a large system of non-linear equations is extremely complicated because of
  - the mathematical techniques involved
  - the computational power needed

The Linearized system of equations (II)

- A first-order Taylor expansion around a convenient set $\mathbf{x}_0$ of starting values (catalog) of the unknown parameters $\mathbf{x} \equiv \{\mathbf{x}^{\mathrm{S}}, \mathbf{x}^{\mathrm{A}}, \mathbf{x}^{\mathrm{C}}, \mathbf{x}^{\mathrm{G}}\}$ linearizes the observation equations and the equation system

$$-\sin\phi_{\mathrm{calc}}\,\delta\phi = \sum_{\mathrm{Source}} \left.\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}^{\mathrm{S}}}\right|_{\mathbf{x_0}} \delta\mathrm{x}^{\mathrm{S}} + \sum_{\mathrm{Attitude}} \left.\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}^{\mathrm{A}}}\right|_{\mathbf{x_0}} \delta\mathrm{x}^{\mathrm{A}} + \sum_{\mathrm{Cal}} \left.\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}^{\mathrm{C}}}\right|_{\mathbf{x_0}} \delta\mathrm{x}^{\mathrm{C}} + \sum_{\mathrm{Global}} \left.\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}^{\mathrm{G}}}\right|_{\mathbf{x_0}} \delta\mathrm{x}^{\mathrm{G}}$$

$$\delta\phi = \phi_{\mathrm{obs}} - \phi_{\mathrm{calc}}$$

$$\delta\mathbf{x} = \mathbf{x}_{\mathrm{true}} - \mathbf{x}_0$$

$$\phi_{\mathrm{calc}} = F(\mathbf{x}_0)$$

- The new unknowns are the corrections to the catalog values. Their estimation $\bar{\delta\mathbf{x}}$ gives

$$\mathbf{x}_{\mathrm{true}} \simeq \bar{\mathbf{x}} = \mathbf{x}_0 + \bar{\delta\mathbf{x}}$$

- The resulting $m \times n$ system of equations is:
    - sparse $\Rightarrow$ #of $(a_{ij} \neq 0) \ll m \times n$
    - overdetermined $\Rightarrow$ $n \ll m$

Solving the Equation System

The NON-feasibility of Direct methods

- Linear System of Equation: $\mathbf{b} = A\mathbf{x}$, sparse, overdetermined

$$\mathbf{x} = \left(A^T A\right)^{-1} A^T \mathbf{b}$$

- Direct methods: needed operations$\propto N_{\mathrm{unk}}^3 \sim 2 \cdot 10^{26}$

Most powerful supercomputer as of November 2018: Summit, DOE/SC/Oak Ridge National Laboratory, USA, 143,500 TFlop/s

Time needed: 47 years!

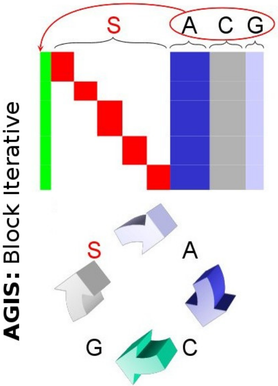
Solving the Equation System

The AGIS approach

- Linear System of Equation:
  $\mathbf{b} = A\mathbf{x}$, sparse, overdetermined

$$\mathbf{x} = \left(A^T A\right)^{-1} A^T \mathbf{b}$$

- AGIS approach: block iterative
  - each "block of unknowns" is solved separately assuming the others as known
  - after all blocks are solved, the process is repeated iteratively



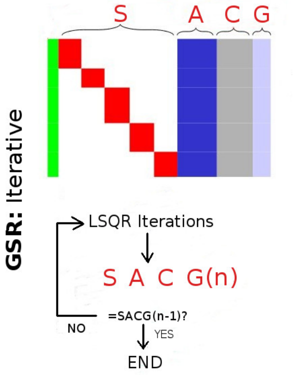
Solving the Equation System

The GSR approach

- Linear System of Equation:
  $\mathbf{b} = A\mathbf{x}$, sparse, overdetermined

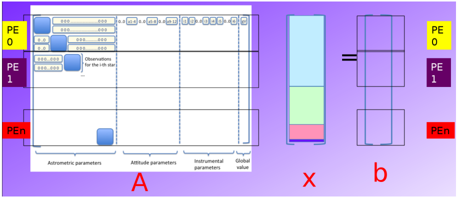
$$\mathbf{x} = \left(A^T A\right)^{-1} A^T \mathbf{b}$$

- GSR approach: iterative
  - complete system solved with an iterative algorithm (LSQR)
  - if needed, the process is repeated using the previous solution as starting values



The need for HPC parallelization

- Contrary to the Block Iterative one, the Iterative approach needs “non-embarrassingly” parallel techniques
- This called for using:
  - C+MPI+OMP language for the Solver module
  - HPC-dedicated hardware



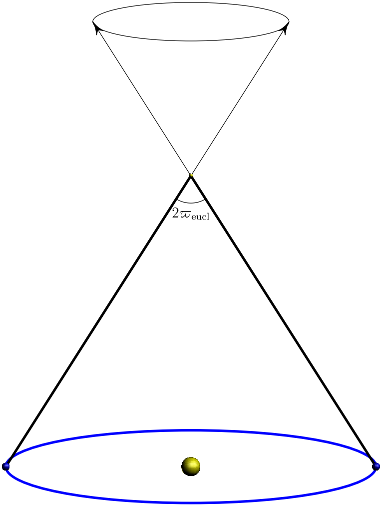
The problem of the variance-covariance matrix

- An estimation of the errors on the system unknowns can be obtained by computing its variance-covariance matrix $S(\mathbf{x})$:

$$\begin{aligned} \mathbf{x} &= \left(A^T A\right)^{-1} A^T \mathbf{b} \\ A^{-g} &= \left(A^T A\right)^{-1} A^T \\ S(\mathbf{x}) &= A^{-g} \left(A^{-g}\right)^T \end{aligned}$$

- The evaluation of $S(\mathbf{x})$ has the same computational complexity of the finding of A^{-g}
- It is not clear how block-iterative methods like AGIS can find the covariances among the unknowns of different blocks
- Iterative methods like the LSQR algorithm adopted by GSR can in principle solve the complete variance-covariance matrix

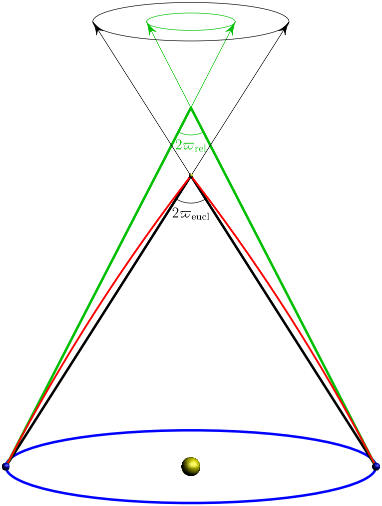
Correlations among different unknowns



Correlations among:

- parallaxes
- Basic Angle variations (Γ)
- Parametrized Post-Newtonian parameter γ

Correlations among different unknowns



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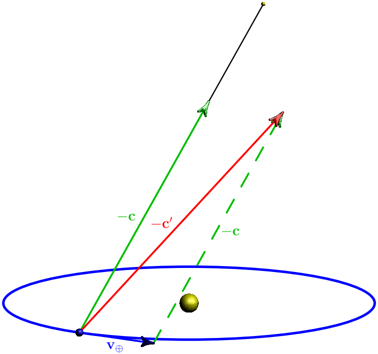
Outline

Light propagation: aberration

- Aberration can be foreseen already in classical astrometry; its natural framework, however, is that of Special (and General) Relativity.
- Classically, it comes from the idea of vectorial **velocity** composition between the velocity of light and that of the observer



$$-\mathbf{c}' = \mathbf{v}_\oplus - \mathbf{c}$$

Light propagation: bending of light

Light bending in Newtonian physics

Equivalence Principle ($m_i = m_g$) + Particle theory of light
= **Light deflection!**

$$\mathbf{a} = \lim_{m_g \to 0} -G \frac{M_g m_g}{m_i r^2} = -G \frac{M_g}{r^2} \quad \Rightarrow \quad \Delta\theta = 2 \frac{GM}{c^2 r}$$

- Newton already argued its existence in the framework of its particle theory of light
- Concept of "Dark star" (Black hole) by John Michell (1783) and Henry Cavendish (1784)
- Fully computed by Soldner (1801)
- No deflection in the wave theory of light!

Light propagation: bending of light

Light bending in Newtonian physics

Equivalence Principle ($m_i = m_g$) + Particle theory of light
= **Light deflection!**

$$\mathbf{a} = \lim_{m_g \to 0} -G\frac{M_g m_g}{m_i r^2} = -G\frac{M_g}{r^2} \quad \Rightarrow \quad \Delta\theta = 2\frac{GM}{c^2 r}$$

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Important change!

Apparent direction of the star given by the photon's **velocity**, not by its position vector ($\mathbf{r}' \to \mathbf{v}$).

$$\cos\psi_{\mathbf{v},\hat{\mathbf{X}}} = \frac{\mathbf{v}\cdot\hat{\mathbf{X}}}{v}$$

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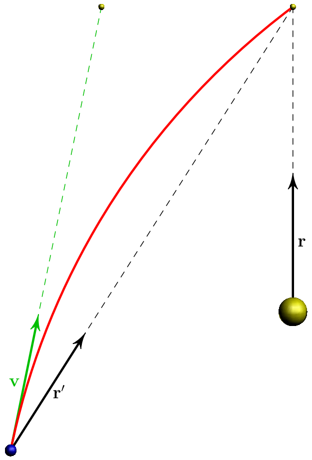
- **Can be included** in the "standard way" **as a "correction"**.



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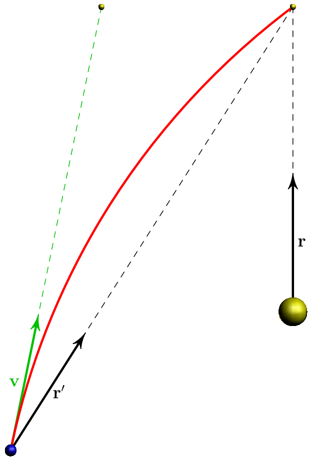

Summing thins up: the need for a General Relativistic approach

Summarizing, we need to know:

- the transformation laws between the inertial reference system and that of the observer;
- the laws of light propagation.

General Relativity changes both things!

Unnatural and Impractical to manage this as corrections to models of classical astroemetry; as accuracy increases more and more "effects" come into play (planetary deflection, quadrupole effects, gravitomagnetic effects, . . .)

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Outline

What is General Relativity about? (1)

General Relativity is a **Theory of gravity**

- Gravity described in a 4D manifold whose coordinates (x^0, x^1, x^2, x^3) tag the **events of the spacetime**.
- 4D distances among the events can be computed using the **metric tensor** $g_{\mu\nu}$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

- Gravity determines the geometry of the manifold, and thus $g_{\mu\nu}$.
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What is General Relativity about? (2)

Field equations $\nabla^2 V = 4\pi G\rho \quad \rightarrow \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

Equations of motion $-\boldsymbol{\nabla} V = \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} \quad \rightarrow \quad -\Gamma^{\lambda}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}s^2}$

- The **stress-energy tensor** $T_{\mu\nu}$ is the relativistic counterpart of the mass density ρ.
- The quantities $R_{\mu\nu}$, R, and $\Gamma^{\lambda}_{\mu\nu}$ are defined in terms of the derivatives of the **metric tensor** $g_{\mu\nu}$, which plays the role of the potential V.
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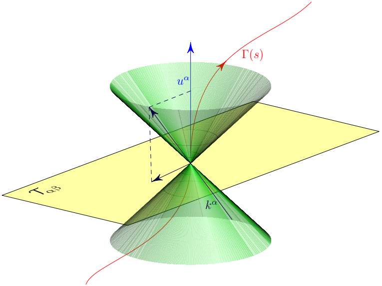
Theory of Measurements for the Impatient

- A physical observer is identified by his/her worldline Γ, whose four-velocity is $u^\alpha \equiv \dot{\Gamma}$ ($\Rightarrow$the parameter s of Γ represents the proper time of the observer).
- $\forall$ point $\in \Gamma$, u^α splits the space-time into a 1D subspace $\| u^\alpha$ and a 3D subspace $\perp u^\alpha$, which are the *time direction* and the *space* relative to the observer u^α respectively.



$$\mathscr{P}_{\alpha\beta} = -u_\alpha u_\beta \qquad \text{Parallel projector}$$

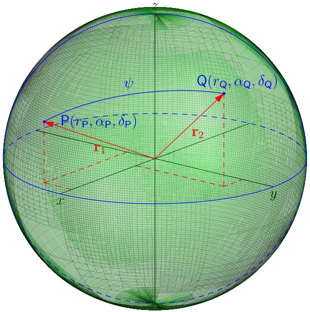
$$\mathscr{T}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \qquad \text{Transverse projector}$$

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= -c^2 \underbrace{\left(-c^{-2}\mathscr{P}_{\alpha\beta}dx^\alpha dx^\beta\right)}_{dt_u^2} + \underbrace{\left(\mathscr{T}_{\alpha\beta}dx^\alpha dx^\beta\right)}_{dl_u^2} \end{aligned}$$

Basic observable: arc

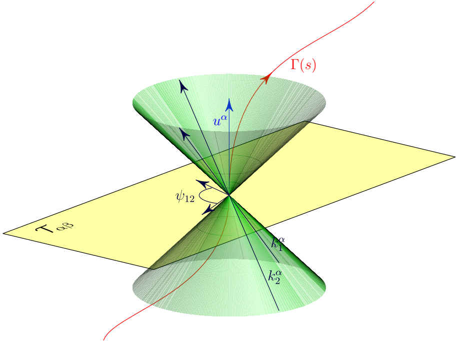
Classical ($\mathbf{r} \to \mathbf{v}$)

$$\cos\psi = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}$$



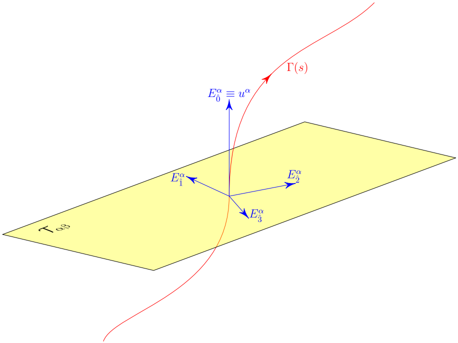
General Relativity ($k^\alpha = \dfrac{\mathrm{d}x^\alpha}{\mathrm{d}s} \to \mathbf{v}$)

$$\cos\psi = \frac{\mathscr{T}_{\alpha\beta}\, k_1^\alpha\, k_2^\beta}{\sqrt{\mathscr{T}_{\alpha\beta}\, k_1^\alpha\, k_1^\beta}\sqrt{\mathscr{T}_{\alpha\beta}\, k_2^\alpha\, k_2^\beta}}$$



Relativistic reference systems: tetrads

- Equivalence Principle $\Rightarrow$ for any u^μ it is always possible to find a locally inertial reference system where Special Relativity holds.
- This reference system is called **tetrad** $\left(E^\alpha_{\hat{0}}, E^\alpha_{\hat{1}}, E^\alpha_{\hat{2}}, E^\alpha_{\hat{3}}\right)$ and can be constructed at any point P by imposing the conditions
  1. $E^\alpha_{\hat{0}} \equiv u^\alpha$
  2. $\left(g_{\alpha\beta} E^\alpha_{\hat{a}} E^\alpha_{\hat{b}}\right)_P = \eta_{\hat{a}\hat{b}}$



- This implies that a tetrad is defined except for three arbitrary spatial rotations

Relativistic reference systems: tetrads

Tetrads can be used when the modeling of a single direction wrt a reference frame is needed (instead, e.g. of an arc).

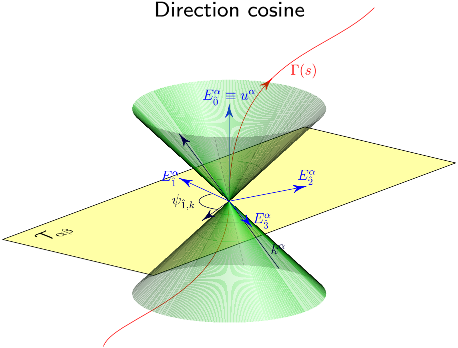
$$\cos\psi_{k,\hat{a}} = \frac{\mathscr{T}_{\alpha\beta}\, k^{\alpha} E^{\beta}_{\hat{a}}}{\sqrt{\mathscr{T}_{\alpha\beta}\, k^{\alpha} k^{\beta}}}$$

$$\downarrow$$

$$\cos\psi_{\mathbf{r}',\hat{\mathbf{X}}} = \frac{\mathbf{r}'\cdot\hat{\mathbf{X}}}{r'}$$

$$\left(\mathbf{r}' \to \mathbf{v}\right)$$

Direction cosine



Summing things up

The **three steps** to build a relativistic model for an astrometric observable from scratch are:

1. **Field equations**
   - in the chosen coordinate system (gauge), find the appropriate stress-energy tensor $T_{\alpha\beta}$ for your physical situation and solve the field equations to get the metric tensor $g_{\alpha\beta}$;
2. **Geodesic equations**
   - use the metric tensor $g_{\alpha\beta}$ to write and integrate the geodesic equations for the photons (null geodesics) at least down to first degree, i.e. find k^{α};
3. **The observer**
   - find u^{α} (barycentric motion) and/or the tetrad $E^{\alpha}_{\hat{a}}$ of the observer and write the equations for your observable.

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Outline

General procedure

The satellite tetrad can be built with a series of relativistically consistent transformations, starting from the BCRS reference system:

1. A shift of the origin from that of the BCRS to the barycenter of the satellite (BCRS $\rightarrow$ *Local* BCRS, $\boldsymbol{\lambda}_{\hat{\alpha}}$)
2. A special relativistic boost of the Local BCRS to take into account of the satellite velocity (Local BCRS $\rightarrow$ *Boosted tetrad* $\tilde{\boldsymbol{\lambda}}_{\hat{\alpha}}$)
3. A purely Euclidean rotation of the spatial axes of the Boosted tetrad to the instantaneous satellite attitude (Boosted tetrad $\rightarrow$ *Satellite Reference System*/SRS $\boldsymbol{E}_{\hat{\alpha}}$)

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The local BCRS reference system

By definition, for this tetrad the time axis belongs to the same congruence of the BCRS. In other words, $\boldsymbol{E}_{\hat{0}}^{\mathrm{BCRS}} \parallel \boldsymbol{E}_{\hat{0}}^{\mathrm{LBCRS}}$. Finding this expression is easy, once we recall that at the origin O of the BCRS it is $E_{\hat{0}\mathrm{BCRS}}^{\alpha} \equiv u_{\mathrm{O}}^{\alpha} = \delta_0^{\alpha}$.

- "Parallel" means "proportional", therefore at the origin O′ of the Local BCRS $u_{\mathrm{O}'}^{\alpha} = C\, u_{\mathrm{O}}^{\alpha} = C\, \delta_0^{\alpha}$.
- Moreover, $u_{\mathrm{O}'}^{\alpha}$ must be normalized, i.e. $u_{\mathrm{O}'}^{\alpha}(u_{\mathrm{O}'})_{\alpha} = -1$ (time-like four-vector, but watch the signature). This implies $C = (-g_{00})^{-1/2}$.
- Thus, in the case of the BCRS metric, the temporal axis of the Local BCRS tetrad is

$$\lambda_{\hat{0}}^{\alpha} \equiv u_{\mathrm{O}'}^{\alpha} = \left(1 + \frac{1}{2} h_{00}\right) \delta_0^{\alpha} + \mathscr{O}(h^2)$$

- The expressions of three spatial axes, instead, can be found by imposing the orthonormality condition $\left(g_{\alpha\beta} E_{\hat{a}}^{\alpha} E_{\hat{b}}^{\alpha}\right)_{\mathrm{O}'} = \eta_{\hat{a}\hat{b}}$:

$$\lambda_{\hat{a}}^{\alpha} = h_{0a}\delta_0^{\alpha} + \left(1 - \frac{1}{2} h_{00}\right) \delta_a^{\alpha} + \mathscr{O}(h^2)$$

The boosted reference system

This reference system is the tetrad adapted to the motion of the satellite's barycenter. But how do we *model* this motion in a GR-consistent way?

- First of all, as usual, we use coordinate representation

$$\tilde{\boldsymbol{\lambda}}_{\hat{0}} = \boldsymbol{u}_{\mathrm{s}} = T_{\mathrm{s}} \{1, \beta_x, \beta_y, \beta_z\}$$

  where β depends on the coordinate 3-velocity; $\beta_i = v_i/c = \mathrm{d}x/c\mathrm{d}t$, $T_{\mathrm{s}} \simeq 1 + h_{00} + \beta^2/2$, and $\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2$.
- This 4-velocity allows to find the $\gamma = u_{\mathrm{s}}^{\alpha} (u_{\mathrm{O}'})_{\alpha}$ factor of the Lorentz transformation (boost) between the Local BCRS and the boosted tetrad.

The boosted reference system

Lorentz-transforming the spatial axes of the Local BCRS finally gives the complete boosted tetrad.

$$\tilde{\boldsymbol{\lambda}}_{\hat{a}} = \boldsymbol{\lambda}_{\hat{a}} + \begin{pmatrix} \beta_x \left(1 + \frac{3}{2} h_{00} + \frac{1}{2}\beta^2\right) & \frac{1}{2}\beta_x^2 & \frac{1}{2}\beta_x\beta_y & \frac{1}{2}\beta_x\beta_z \\ \beta_y \left(1 + \frac{3}{2} h_{00} + \frac{1}{2}\beta^2\right) & \frac{1}{2}\beta_x\beta_y & \frac{1}{2}\beta_y^2 & \frac{1}{2}\beta_y\beta_z \\ \beta_z \left(1 + \frac{3}{2} h_{00} + \frac{1}{2}\beta^2\right) & \frac{1}{2}\beta_x\beta_z & \frac{1}{2}\beta_y\beta_z & \frac{1}{2}\beta_z^2 \end{pmatrix}$$

The Satellite Reference System (SRS)

The boosted reference system (tetrad) and the SRS are just connected by a spatial (3D) rotation. Namely, the temporal axis remains the same (the motion of the barycenter is the same) whereas the spatial axes change their orientation according to the satellite attitude.

$$\begin{pmatrix} \boldsymbol{E}_{\hat{1}} \\ \boldsymbol{E}_{\hat{2}} \\ \boldsymbol{E}_{\hat{3}} \end{pmatrix} = R \begin{pmatrix} \tilde{\boldsymbol{\lambda}}_{\hat{1}} \\ \tilde{\boldsymbol{\lambda}}_{\hat{2}} \\ \tilde{\boldsymbol{\lambda}}_{\hat{3}} \end{pmatrix}.$$

The rotation matrix R (*attitude matrix*) can be written with any suitable parametrization (e.g. Euler angles, quaternions, MRP, etc.) Euclidean transformation but transforms a 4-vector into another 4-vector!

Outline

Numerical challenges

- Solving in a reasonable amount of time and in the most efficient way such large systems of equations requires non-trivial adjustment of the existing algorithms (i.e. not just the standard LSQR)
- A fundamental problem in data reduction is to have a reliable estimation of errors, that means of variances and covariances. In the case of the Gaia mission this cannot be taken for granted.
  - Block-Iterative techniques are likely to have difficulties with covariances between different blocks
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What is the correct theory of gravity?

- Not surprisingly, there exist many alternatives to GR.
- The PPN formalism represents a powerful formalism to model the prediction of different theories of gravity (Lecture by CLPL).
- By writing the observation equation in the PPN framework, we can regard the PPN parameters as unknowns (Global parameters), thus casting the Global Sphere Reconstruction problem as a giant test of Gravity theory.
- The most convenient parameter to estimate is the PPN γ (not to be confused with the Lorentz factor!).
- Current best estimation from the Cassini experiment ($\sigma_\gamma \sim 10^{-5}$)
- Gaia should be able to improve this estimation by an order of magnitude.

Quadrupole effects on light deflection

- Light deflection in the Solar System was always considered in the case of spherically symmetric massive bodies.
- The expected Gaia accuracy makes it feasible to attempt the first detection of the deviation from a symmetric pattern caused by a non-symmetric gravity field. (At least for compact objects.)
- In particular, the easiest possibility seems to be that of the Jupiter quadrupole, which amounts to about 240μas at the planetary border. (But decreases as d^{-3})
- This effect might be estimated by Gaia again as a by-product of the Global Sphere Reconstruction, or (see B.B. lecture) through techniques of differential astrometry.

The scientific point of view in summary

- The challenge of *defining* and *solving* precise observations assembled in such a large system of equations is of huge scientific interest *per se*
  - calls for the determination of the best way to model the observations
  - helps to develop new perspectives on the reduction of global astrometric data
  - computationally intensive task (parallelization)
  - the problem of the variance-covariance matrix determination is still being investigated in the literature
- The determination of a full-sky “pseudo” inertial reference frame is a problem of fundamental physics
- An order of magnitude improvement of light deflection test for competing theories of Gravity in the PPN framework

Conclusions

- Solving the problem of mathematical modeling of astrometric observations requires the knowledge of:
  1. the transformation laws between different reference systems;
  2. the laws of light propagation.
- A general relativistic treatment of this problems requires three steps:
  1. solve the field **Field equations** and the metric tensor $g_{\alpha\beta}$ ($\rightarrow$ define the "inertial" reference system of the final unknowns);
  2. solve the **Geodesic equations**and find k^{α} (the "velocity" of the photons);
  3. find the **tetrad of the observer** $E^{\alpha}_{\hat{a}}$ and write the equations for your observable.
- This problem has been formulated in several different ways, and some of them are going to be applied to the case of Gaia (see S.B.'s lecture).
- The accuracy of this mission requires a careful comparison of the different models both from the analytical and the numerical point of view.

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- This problem has been formulated in several different ways, and some of them are going to be applied to the case of Gaia (see S.B.'s lecture).
- The accuracy of this mission requires a careful comparison of the different models both from the analytical and the numerical point of view.

Conclusions

- Solving the problem of mathematical modeling of astrometric observations requires the knowledge of:
    1. the transformation laws between different reference systems;
    2. the laws of light propagation.
- A general relativistic treatment of this problems requires three steps:
    1. solve the field **Field equations** and the metric tensor $g_{\alpha\beta}$ ($\rightarrow$ define the "inertial" reference system of the final unknowns);
    2. solve the **Geodesic equations**and find k^{α} (the "velocity" of the photons);
    3. find the **tetrad of the observer** $E^{\alpha}_{\hat{a}}$ and write the equations for your observable.
- This problem has been formulated in several different ways, and some of them are going to be applied to the case of Gaia (see S.B.'s lecture).
- The accuracy of this mission requires a careful comparison of the different models both from the analytical and the numerical point of view.