Some basics General Relativity for the beginner

How to construct an astronomical reference system

C. Le Poncin-Lafitte - LNE-SYRTE, Paris Obs.

cnrs
SORBONNE UNIVERSITÉS
UPMC SORBONNE UNIVERSITÉS
LNE Le progrès, une passion à partager
l'Observatoire de Paris SYRTE
Systèmes de Référence Temps-Espace

Motivations

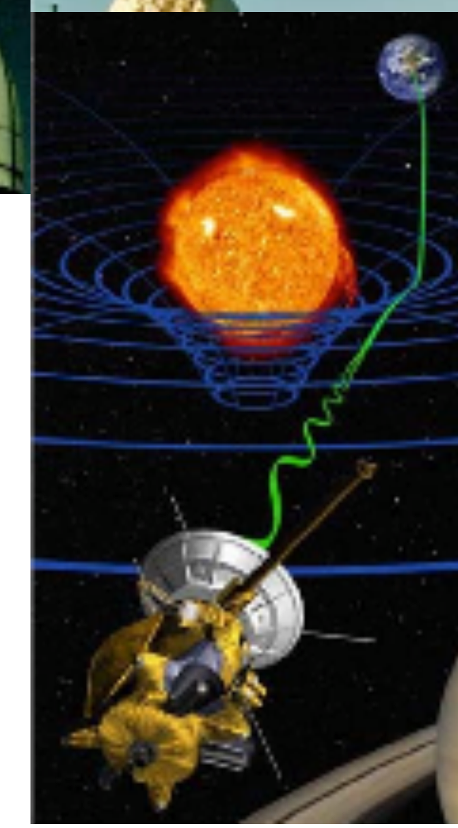

Ground & space geodesy accuracy is increasing:

LLR & SLR ⟶ From cm to mm
GALILEO


Gravity Probe A to ACES/Pharao ⟶ factor 80 on Grav. Redshift

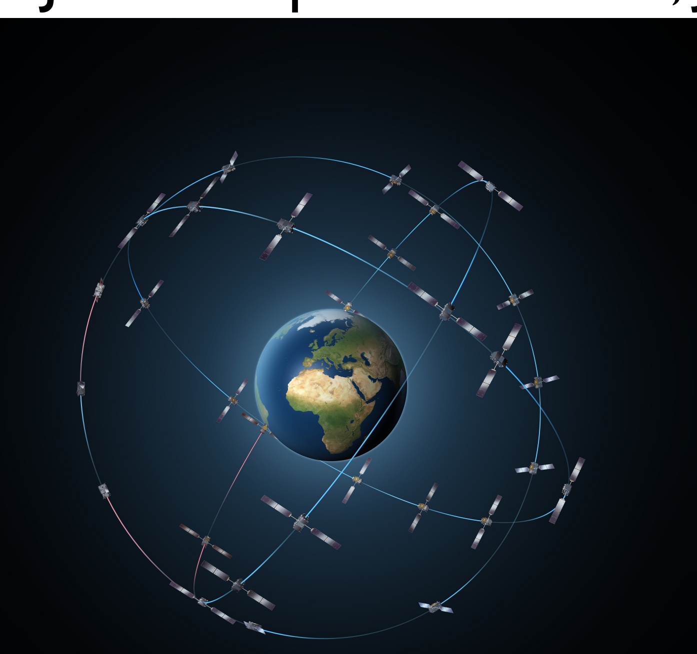
Ground & space astrometry:

Gaia, Gravity ⟶ from milli to micro-arcsecond



Navigation of interplanetary probes :

Cassini Experiment, use of Ka Band
MORE Experiment on BepiColombo ⟶ factor 10 on Doppler
JUNO Experiment 2016, JUICE towards 2030




Need to describe light propagation and dynamics in a relativistic framework

- How to solve the field equation
- Need to introduce new tools

and define properly the observables !

Special Relativity in some words…

Inertial frame and Principle of Relativity

Let us suppose 2 frames S ans S' with coordinates

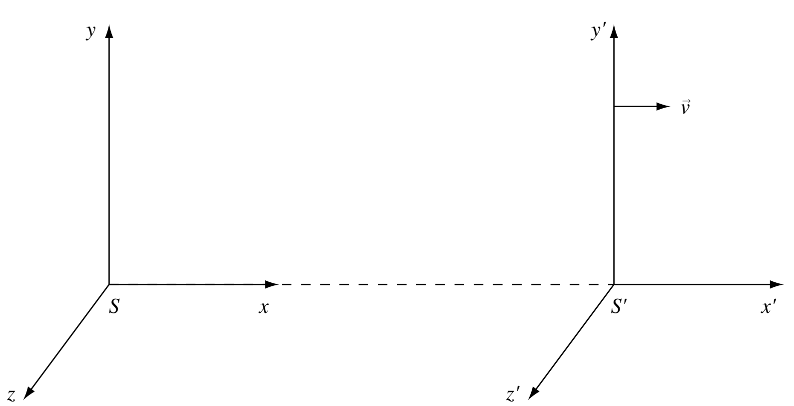
$$S = (ct, x, y, z)$$
$$S' = (ct', x', y', z')$$

Imagine now a free particule, P, existing somewhere. How to represent P in S and S' ?

If S and S' are inertial, the Newton First Law holds. In S, we have $\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0.$

→ Free particule at rest or linear motion

We have the same equation for this particule in S' with prime coordinates..



If now, S' is in motion with respect to S, in x-direction with constant velocity. Suppose that at t=t'=0, S and S' coincide.

How to link the coordinates of P in S and S' ?

Special Relativity in some words…

Inertial frame and Principle of Relativity

First key point of Special Relativity : Principle of Relativity.

Physics must be the same in all inertial frames…

One has to imagine the most simple linear transformation

$$\begin{cases} t' = At + Bx, \\ x' = Dt + Ex, \\ y' = y, \\ z' = z. \end{cases}$$

One has to find A, B, D and E…

First, we now the motion of S' as constant in x-direction so

$$\begin{cases} t' = At + Bx, \\ x' = A(x - vt), \\ y' = y, \\ z' = z. \end{cases}$$

Second key point of Special Relativity : the speed of light is constant in inertial frame.

Imagine a photon emitted from the coincident S and S' at t=t'=0 and travelling in an arbitrary direction. Time and space coordinates of that photon in each frame must satisfy

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0.$$

Special Relativity in some words…

Inertial frame, Principle of Relativity, constant speed of light

Let us now combine

$$\begin{aligned} t' &= At + Bx, \\ x' &= A(x - vt), \\ y' &= y, \\ z' &= z. \end{aligned}$$

and $c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0.$

$$\begin{aligned} ct' &= \gamma(ct - \beta x), \\ x' &= \gamma(x - \beta ct), \\ y' &= y, \\ z' &= z, \end{aligned}$$

with $\gamma = (1 - \beta^2)^{-1/2}$, $\beta = v/c$

Lorentz transform…

Notion of interval

Consider now two events in spacetime, A and B. We can define the (squared) interval as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

with $dt = t_B - t_A$, $dx = x_B - x_A$, $dy = y_B - y_A$, $dz = z_B - z_A$

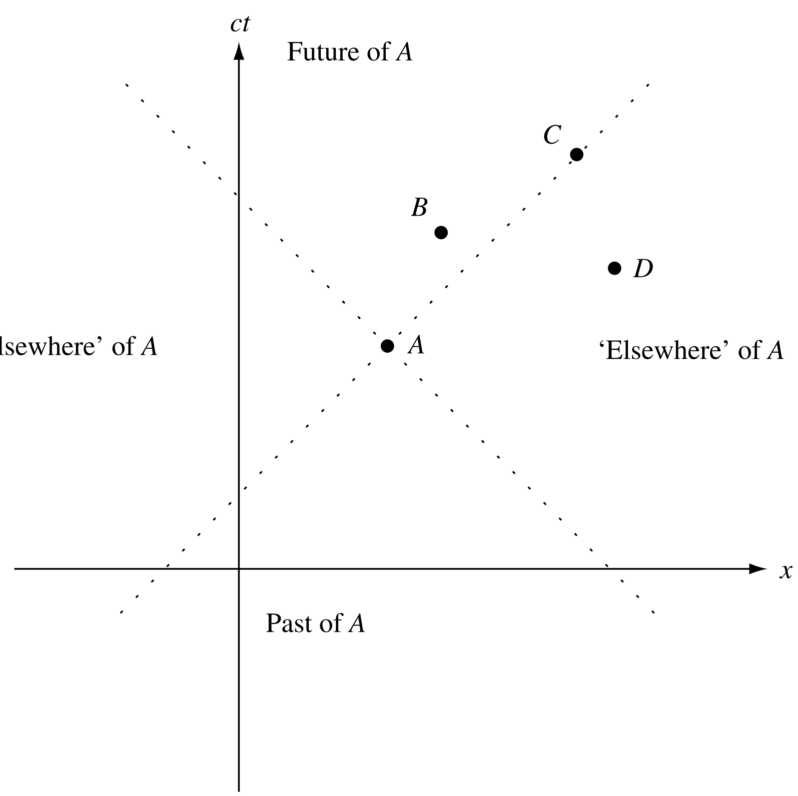
It is straightforward to show that ds^2 is conserved under any Lorentz transformation.

Let us finally introduce the Minkowski tensor as $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$

Special Relativity in some words...

Interval and lightcone

Let us consider a point-event A and represent it in a Minkowski diagram



$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2$$

Let us consider others point-events (B, C, D). If

$ds^2 = 0 \rightarrow c^2 dt^2 = dx^2$ Lightlike or null

$ds^2 > 0 \rightarrow c^2 dt^2 > dx^2$ Timelike

$ds^2 < 0 \rightarrow c^2 dt^2 < dx^2$ Spacelike

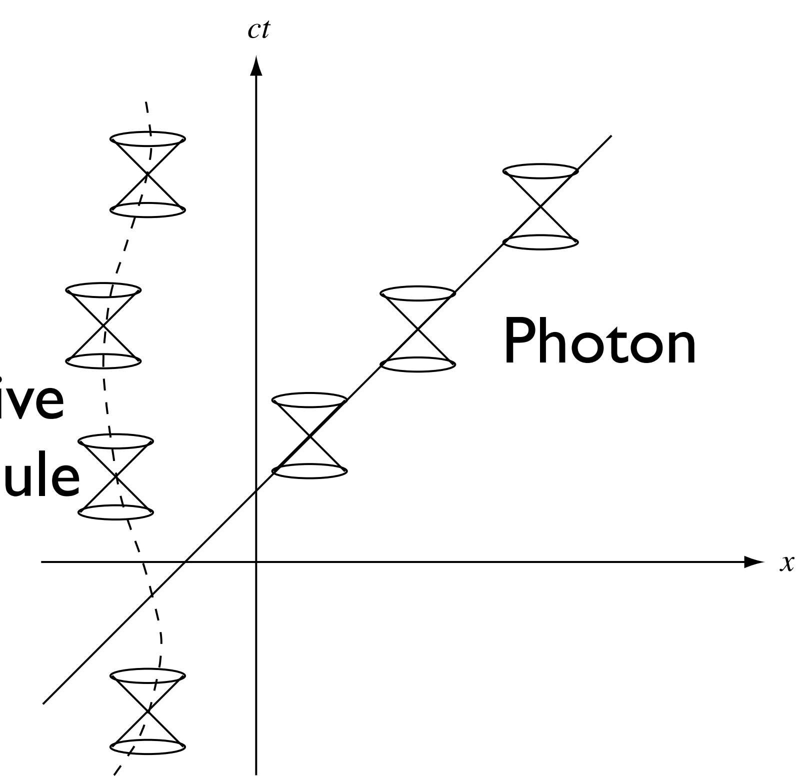
Particule worldlines and proper time

for $ds^2 > 0$, the interval is timelike;

for $ds^2 = 0$, the interval is null or lightlike;

for $ds^2 < 0$, the interval is spacelike.

Definition of the proper time : $c^2 d\tau^2 = ds^2$



Special Relativity in some words…

Lorentz group

Flat (or Minkowski) spacetime of Special Relativity is a fixed four-dimensional pseudo-Euclidean manifold. It exists a privileged class of Cartesian coordinate system (ct, x, y, z) covering the whole spacetime where the (squared) interval takes, at every point-event, the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \qquad [\eta_{\mu\nu}] = diag(1, -1, -1, -1)$$

Transforming to a different Cartesian inertial frame corresponds to a new coordinates system (ct', x', y', z') and must satisfy

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx'^\mu dx'^\nu \longrightarrow \eta_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \eta_{\rho\sigma}$$

Thus the transformation between 2 inertial frames must be linear $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$

2 constants $a^\mu \equiv 0$

where we have $[\Lambda^\mu{}_\nu] = \left[\frac{\partial x'^\mu}{\partial x^\nu}\right] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

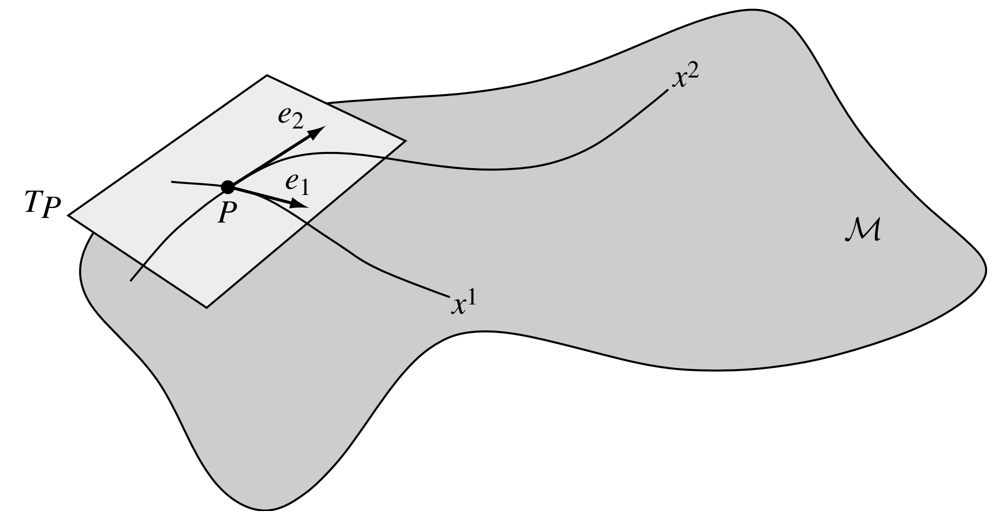
with the properties $\Lambda^\nu_\mu = \eta_{\mu\rho} \eta^{\nu\sigma} \Lambda^\rho_\sigma$ and $\Lambda^\mu_\nu \Lambda^\sigma_\mu = \delta^\sigma_\nu$

Towards General Relativity

Notion of curvature…

In Special Relativity, the metric tensor corresponds to flat space time $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

Let us consider a more general manifold.
At each point P, one can define a coordinate basis



$$\boldsymbol{e}_a = \lim_{\delta x^a \to 0} \frac{\delta \boldsymbol{s}}{\delta x^a},$$

where $\delta \boldsymbol{s}$ is the infinitesimal vector displacement between P and a nearby point Q

$$d\boldsymbol{s} = \boldsymbol{e}_a(x)\, dx^a$$

$$ds^2 = (e_a dx^a)(e_b dx^b) = (e_a e_b) dx^a dx^b$$

More generally speaking, let us define the metric tensor as follow

$$g_{\mu\nu} = e_\mu . e_\nu$$

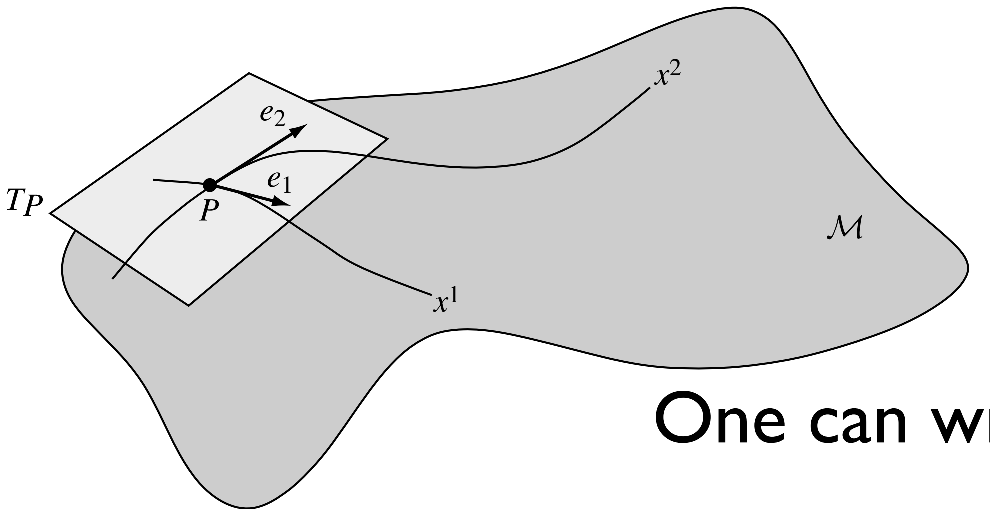
In this case, the interval can be written :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

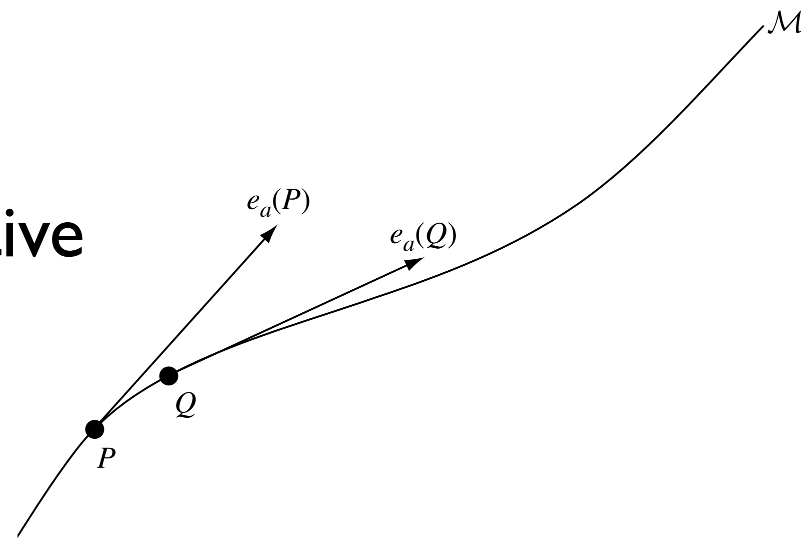
Towards General Relativity

Notion of curvature…

But local vectors at different points P and Q of the manifold lie in different tangent spaces. → NO WAY to add or subtract them…



How to define the derivative of a vector field ?



One can write $\boldsymbol{e}_a(Q) = \boldsymbol{e}_a(P) + \delta \boldsymbol{e}_a$

One has now to calculate the partial derivate $\frac{\partial \boldsymbol{e}_a}{\partial x^c} \equiv \left(\lim_{\delta x^c \to 0} \frac{\delta \boldsymbol{e}_a}{\delta x^c} \right)_{\| T_P}$ and project it into tangent space at P !

It leads to the definition of the affine connection $\boxed{\frac{\partial \boldsymbol{e}_a}{\partial x^c} = \Gamma^b{}_{ac} \boldsymbol{e}_b}$

Since every coordinate basis must be reciprocital → $\boldsymbol{e}^a \cdot \boldsymbol{e}_b = \delta^b_a$

One gets $\partial_c (\boldsymbol{e}^a \cdot \boldsymbol{e}_b) = (\partial_c \boldsymbol{e}^a) \cdot \boldsymbol{e}_b + \boldsymbol{e}^a \cdot (\partial_c \boldsymbol{e}_b) = 0$ → $\boxed{\partial_c \boldsymbol{e}^a = -\Gamma^a{}_{bc} \boldsymbol{e}^b}$

Final definition of the connection $\Gamma^a{}_{bc} = \boldsymbol{e}^a \cdot \frac{\partial \boldsymbol{e}_b}{\partial x^c}$ ⟷ Link with metric tensor ?

Towards General Relativity

Connection versus metric tensor

For simplicity, let us assume that the connection is symmetric (no torsion) : $\Gamma^b_{ac} \equiv \Gamma^b_{ca}$

The metric tensor has been defined as $g_{\mu\nu} = e_\mu . e_\nu$

If we differentiate the metric tensor, what's happen ? $\longrightarrow$ $\partial_c g_{ab} = (\partial_c \boldsymbol{e}_a) \cdot \boldsymbol{e}_b + \boldsymbol{e}_a \cdot (\partial_c \boldsymbol{e}_b)$

But… We have just seen that $\frac{\partial \boldsymbol{e}_a}{\partial x^c} = \Gamma^b{}_{ac} \boldsymbol{e}_b$!!

$$\Longrightarrow \quad \partial_c g_{ab} = \Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad}$$

We can also permute the indice and see what is going on….

$$\partial_b g_{ca} = \Gamma^d{}_{cb} g_{da} + \Gamma^d{}_{ab} g_{cd},$$
$$\partial_a g_{bc} = \Gamma^d{}_{ba} g_{dc} + \Gamma^d{}_{ca} g_{bd}.$$

And finally, forming the combination $\partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}$ and contracting by g^{ea}

$$\boxed{\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})}$$

Towards General Relativity

Geodesics : intrinsic derivative of a vector along a curve

Let us consider a curve C. At any point along C, we have a vector field as $\boldsymbol{v}(u) = v^a(u)\boldsymbol{e}_a(u)$, where $\mathrm{e}_a(u)$ are the coordinate basis at a point on C corresponding to parameter u.

Thus, the derivative of v along C is given by $\frac{d\boldsymbol{v}}{du} = \frac{dv^a}{du}\boldsymbol{e}_a + v^a \frac{d\boldsymbol{e}_a}{du} = \frac{dv^a}{du}\boldsymbol{e}_a + v^a \frac{\partial \boldsymbol{e}_a}{\partial x^c}\frac{dx^c}{du}$,

But we have established before $\frac{\partial \boldsymbol{e}_a}{\partial x^c} = \Gamma^b{}_{ac}\boldsymbol{e}_b$!!!!

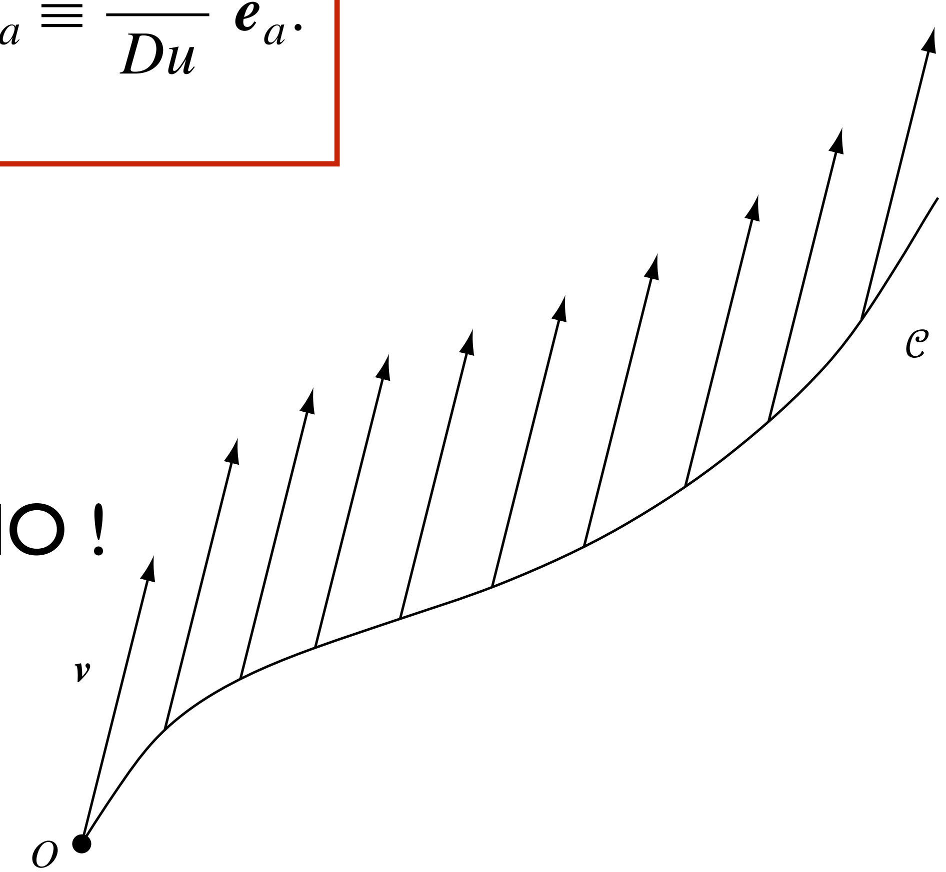
$$\frac{d\boldsymbol{v}}{du} = \left(\frac{dv^a}{du} + \Gamma^a{}_{bc}v^b\frac{dx^c}{du}\right)\boldsymbol{e}_a \equiv \frac{Dv^a}{Du}\,\boldsymbol{e}_a.$$

Geodesics : parallel transport

We want something like $\frac{d\mathbf{v}}{du} = \mathbf{0} \rightarrow \frac{dv^a}{du} = 0$

In pseudo-euclidian space, it works… But in general, NO !

$$\frac{Dv^a}{Du} \equiv \frac{dv^a}{du} + \Gamma^a_{bc}v^b\frac{dx^c}{du} = 0.$$



Towards General Relativity

Equation of geodesics

Let us consider a curve $x^a(u)$ parameterized by some general parameter u and $t^a(u)$ the vector tangent to the curve.

The variation of the tangent vector defines the curve without any doubt.
Let us assume the most simple evolution $\frac{d\mathbf{t}}{du} = \lambda(u)\mathbf{t}$

To be determined !

Using the coordinate basis and the result concerning parallel transport, we must satisfy

$$\frac{Dt^a}{Du} = \frac{dt^a}{du} + \Gamma^a_{bc} t^b \frac{dx^c}{du} = \lambda(u) t^a$$

But $t^a = dx^a/du$

$$\frac{Dt^a}{Du} = \frac{d^2x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = \lambda(u) \frac{dx^a}{du}$$

The question is now to define $\lambda(u) \longrightarrow$ Let choose u as an affine parameter… $\lambda(u) = 0$

Geodesic equations : $$\frac{d^2x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = 0$$

General Relativity

The Equivalence Principle

- 3 facets: Universality of free fall, Local Position/Lorentz Invariance
- very well tested (10^{-13} with Eöt-wash experiments and Lunar Laser Ranging ; 10^{-4} with grav. redshift ; no variation of constants)[1]
- more accurate measurement needed: alternative (string) theories predict violation smaller[2] → MICROSCOPE accuracy 10^{-15}

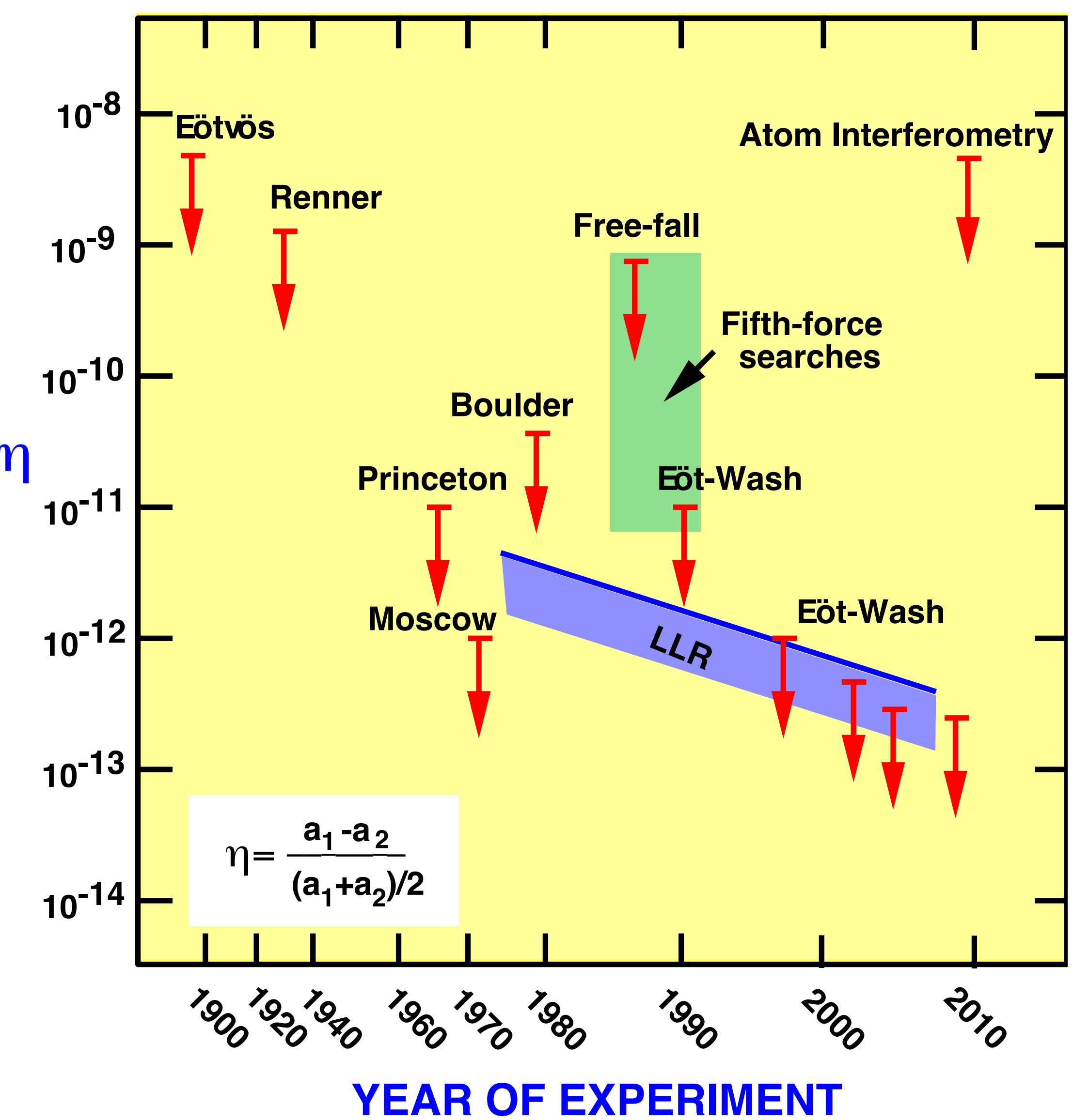
- Gravitation ⇔ space-time curvature (described by a metric $g_{\mu\nu}$)

  Einstein intuition : matter curves spacetime

- free-falling masses follow geodesics of this metric and ideal clocks measure proper time $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Free Fall Experiments

TESTS OF THE WEAK EQUIVALENCE PRINCIPLE



400 CE Ioannes Philiponus: *"...let fall from the same height two weights of which one is many times as heavy as the other .... the difference in time is a very small one"*

1553 Giambattista Benedetti
proposed equality

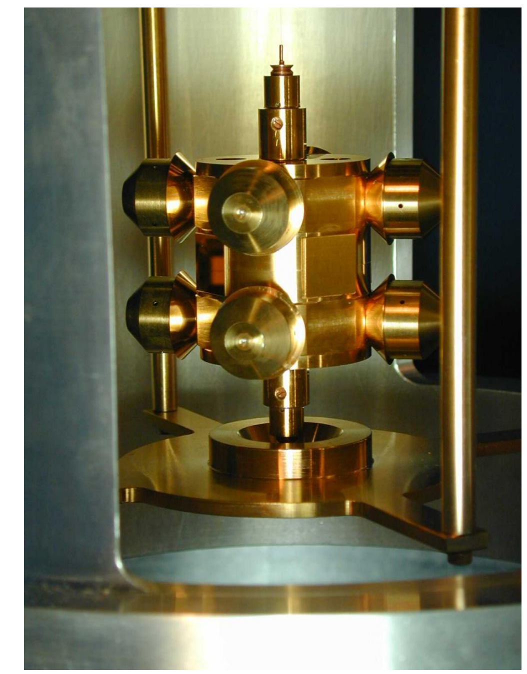
1586 Simon Stevin
experiments

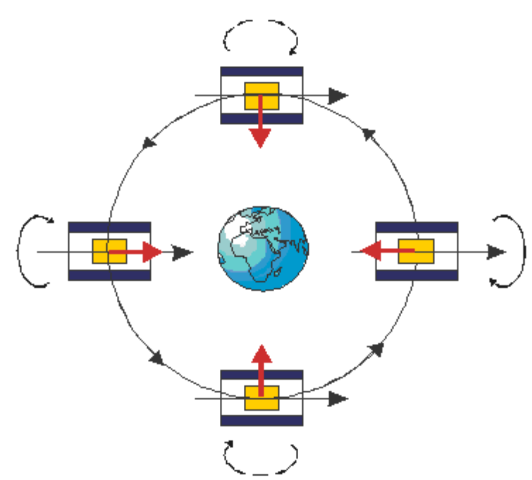
1589-92 Galileo Galilei
Leaning Tower of Pisa?

1670-87 Newton
pendulum experiments

1889, 1908 Baron R. von Eötvös
torsion balance experiments (10^{-9})

1990s UW (Eöt-Wash) *10^{-13}*

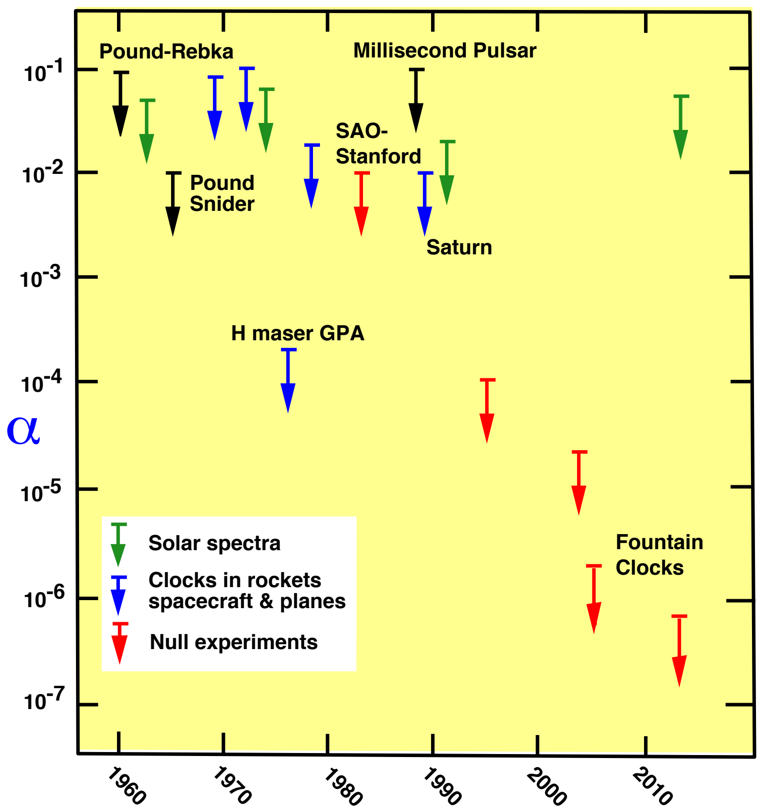




CNES Microscope Mission : 10^{-15}

Local Position Invariance : redshift

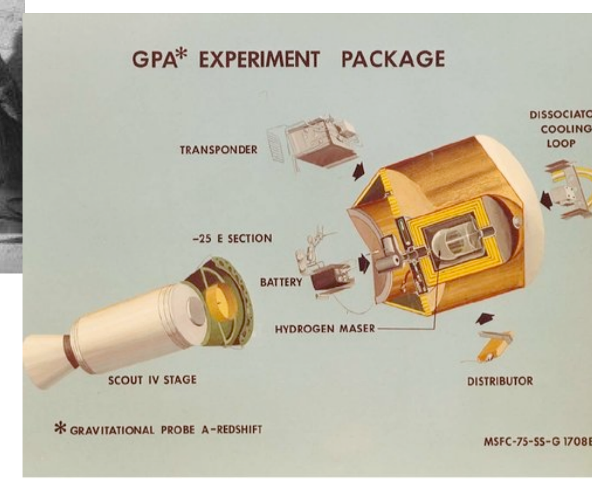
TESTS OF
LOCAL POSITION INVARIANCE



YEAR OF EXPERIMENT

$\Delta\nu/\nu = (1+\alpha)\Delta U/c^2$

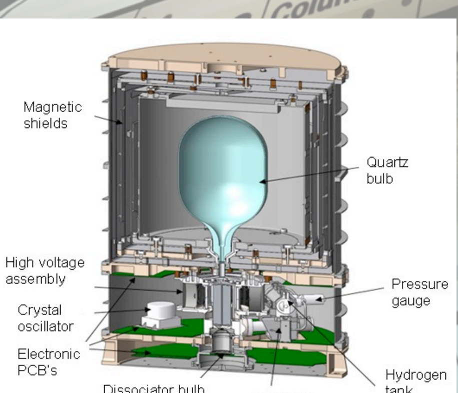
1959 : Pound & Rebka (10%)

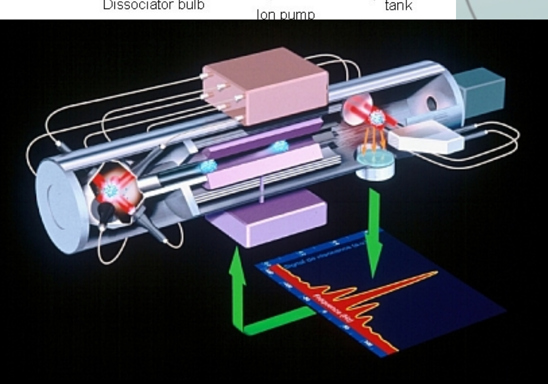


1980 : Gravity Probe A
Vessot (0.01%)

Launch : 1976 with Scout rocket
duration : 1h55mn
where : Wallops Island

Atomic Clock Ensemble in Space (ACES)





Goal :
improving by 35 GPA

To be launched soon

General Relativity

Curvature of a manifold: the Riemann tensor

Let us go back to the covariant derivative of a vector field $\nabla_b v_a = \partial_b v_a - \Gamma^d{}_{ab} v_d$

A second covariant differentiation then yields

$$\begin{aligned}\nabla_c \nabla_b v_a &= \partial_c(\nabla_b v_a) - \Gamma^e{}_{ac} \nabla_b v_e - \Gamma^e{}_{bc} \nabla_e v_a \\ &= \partial_c \partial_b v_a - (\partial_c \Gamma^d{}_{ab}) v_d - \Gamma^d{}_{ab} \partial_c v_d \\ &\quad - \Gamma^e{}_{ac}(\partial_b v_e - \Gamma^d{}_{eb} v_d) - \Gamma^e{}_{bc}(\partial_e v_a - \Gamma^d{}_{ae} v_d)\end{aligned}$$

Swapping indices b and c, we can construct the following tensorial quantity

$$\nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a = R^d{}_{abc} v_d$$

where $$R^d{}_{abc} \equiv \partial_b \Gamma^d{}_{ac} - \partial_c \Gamma^d{}_{ab} + \Gamma^e{}_{ac} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ec}$$

is the Riemann tensor

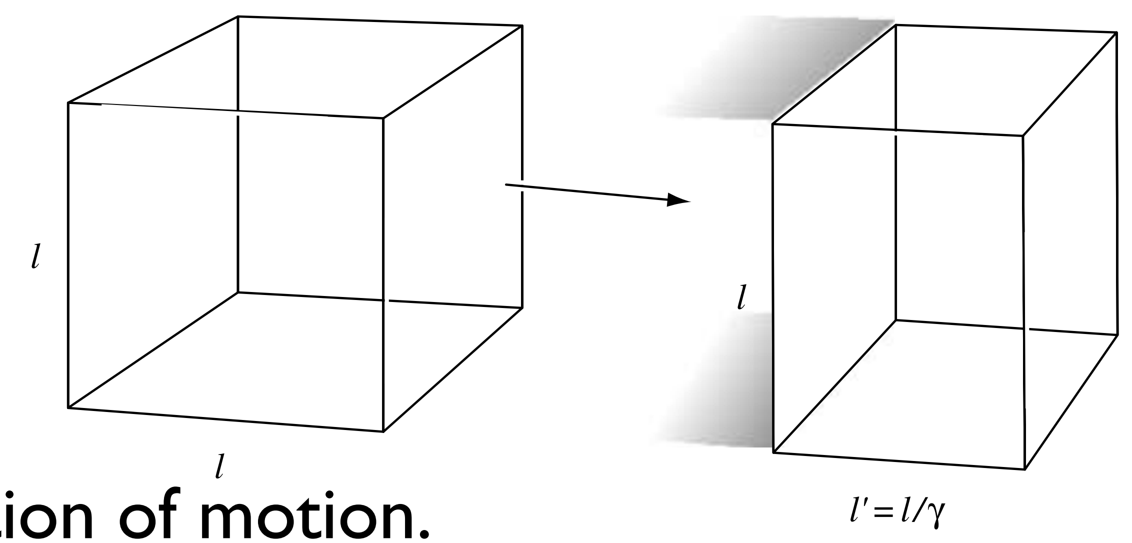
General Relativity

Matter content : the energy-momentum tensor $T_{\mu\nu}$

We need to describe matter content in a covariant way.

Let us consider N dust particules. Let be M the rest mass of all non-interacting particule.

At each event P in spacetime, this dust is fully characterized by giving the matter density and its velocity measured in an inertial frame.



Lorentz contracted in direction of motion

In rest frame, the velocity is null.
The density is $\rho = M \times N$

In other frame, boosted, the volume of the containing the dust is contracted along the direction of motion.
In this frame, we have $N' = \gamma N \qquad M' = \gamma M \qquad \rightarrow \rho' = \gamma^2 \rho$

The matter density is not a *scalar* but *does* transform as the 00-component of a tensor

The most obvious and simple choice is then $\boldsymbol{T}(x) = \rho_0(x)\boldsymbol{u}(x) \otimes \boldsymbol{u}(x)$

General Relativity

Signification of the energy-momentum tensor

T^{00} is the energy density of the particles;
T^{0i} is the energy flux $\times c^{-1}$ in the i-direction;
T^{i0} is the momentum density $\times c$ in the i-direction;
T^{ij} is the rate of flow of the i-component of momentum per unit area in the j-direction.

In an inertial frame, we have

$$T^{00} = \rho u^0 u^0 = \gamma_u^2 \rho c^2,$$

$$T^{0i} = T^{i0} = \rho u^0 u^i = \gamma_u^2 \rho c u^i,$$

$$T^{ij} = \rho u^i u^j = \gamma_u^2 \rho u^i u^j.$$

Equation of motion of the matter content

Let us try an analogy with the conservation of charge : $\partial_\mu j^\mu = 0$

$\longrightarrow$ $\partial_\mu T^{\mu\nu} = 0$ gives directly the equation of motion of the fluid and the equation of continuity

But we have seen that partial derivative are not covariant, we must used in fact the covariant derivative :

$$\nabla_\mu T^{\mu\nu} = 0$$

General Relativity

The Field Equations… (finally !)

One must realized that it is a postulate of Einstein : $K_{\mu\nu} = \kappa T_{\mu\nu}$

$K_{\mu\nu}$ is a 2 rank. Tensor related to the curvature… But how ?

It must have some properties :

- must be symmetric as $T^{\mu\nu}$
- Must satisfy an analog to $\nabla_\mu K^{\mu\nu} = 0$

The most general choice is $K_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu} + \lambda g_{\mu\nu}$

But in order to fulfill all properties, we arrive necessarily to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}$$

What is the value of kappa ? => weak field approximation to recover Newton Gravitation !

$$\longrightarrow \quad \kappa = \frac{8\pi G}{c^4}$$

Towards Relativistic reference frames

Weak Field approximation of General Relativity

Solar System is a weak field $\frac{GM}{c^2 r}$ =10^{-9} Earth, 10^{-6} Sun

⟶ Metric can be expanded in series… $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $|h_{\mu\nu}| << 1$

Lorentz group in this case $h'_{\mu\nu} = \Lambda^{\rho}_{\mu}\Lambda^{\sigma}_{\nu}h_{\mu\nu}$

Infinitesimal coordinate transform can be written $x'^{\mu} = x^{\mu} + \xi^{\mu} \rightarrow \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu}$

The metric can transform as follow

$$g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}g_{\rho\sigma} = (\delta^{\rho}_{\mu} - \partial_{\mu}\xi^{\rho})(\delta^{\sigma}_{\nu} - \partial_{\nu}\xi^{\sigma})(\eta_{\rho\sigma} + h_{\rho\sigma})$$
$$= \eta_{\mu\nu} + h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu},$$

⟶ $h'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}$

The contravariant components of the metric tensor are simply

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \text{with} \quad h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$$

Towards Relativistic reference frames

Linearized Field Equations

We have to expand $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}$

First, the connection… $\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\rho}(\partial_\nu h_{\rho\mu} + \partial_\mu h_{\rho\nu} - \partial_\rho h_{\mu\nu}) = \frac{1}{2}(\partial_\nu h^{\sigma}_{\mu} + \partial_\mu h^{\sigma}_{\nu} - \partial^{\sigma} h_{\mu\nu})$

Then, the Riemann… $R^{\sigma}{}_{\mu\nu\rho} = \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} - \partial_\rho \Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\tau}{}_{\mu\rho}\Gamma^{\sigma}{}_{\tau\nu} - \Gamma^{\tau}{}_{\mu\nu}\Gamma^{\sigma}{}_{\tau\rho}$

$$\longrightarrow \quad \begin{aligned} R^{\sigma}{}_{\mu\nu\rho} &= \tfrac{1}{2}\partial_\nu(\partial_\rho h^{\sigma}_{\mu} + \partial_\mu h^{\sigma}_{\rho} - \partial^{\sigma} h_{\mu\rho}) - \tfrac{1}{2}\partial_\rho(\partial_\nu h^{\sigma}_{\mu} + \partial_\mu h^{\sigma}_{\nu} - \partial^{\sigma} h_{\mu\nu}) \\ &= \tfrac{1}{2}(\partial_\nu \partial_\mu h^{\sigma}_{\rho} + \partial_\rho \partial^{\sigma} h_{\mu\nu} - \partial_\nu \partial^{\sigma} h_{\mu\rho} - \partial_\rho \partial_\mu h^{\sigma}_{\nu}), \end{aligned}$$

Final step… Ricci Tensor and curvature scalar.

$R_{\mu\nu} = \frac{1}{2}(\partial_\nu \partial_\mu h + \Box^2 h_{\mu\nu} - \partial_\nu \partial_\rho h^{\rho}_{\mu} - \partial_\rho \partial_\mu h^{\rho}_{\nu})$ And $R = R^{\mu}_{\mu} = \eta^{\mu\nu} R_{\mu\nu} = \Box^2 h - \partial_\rho \partial_\mu h^{\mu\rho}$

And after a lot of fighting algebra and using $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ and $\bar{h} = -h$

$$\Box^2 \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho \partial_\sigma \bar{h}^{\rho\sigma} - \partial_\nu \partial_\rho \bar{h}^{\rho}_{\mu} - \partial_\mu \partial_\rho \bar{h}^{\rho}_{\nu} = -2\kappa T_{\mu\nu}.$$

Towards Relativistic reference frames

Linearized Field Equations and harmonic jauge

We want to simplify $\Box^2 \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma \bar{h}^{\rho\sigma} - \partial_\nu\partial_\rho \bar{h}^\rho_\mu - \partial_\mu\partial_\rho \bar{h}^\rho_\nu = -2\kappa T_{\mu\nu}.$

Let us try with an infinitesimal change of change of coordinate $h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$

$$\begin{aligned}\bar{h}'^{\mu\rho} &= h'^{\mu\rho} - \tfrac{1}{2}\eta^{\mu\rho}h' \\ &= h^{\mu\rho} - \partial^\mu\xi^\rho - \partial^\rho\xi^\mu - \tfrac{1}{2}\eta^{\mu\rho}(h - 2\partial_\sigma\xi^\sigma) \\ &= \bar{h}^{\mu\rho} - \partial^\mu\xi^\rho - \partial^\rho\xi^\mu + \eta^{\mu\rho}\partial_\sigma\xi^\sigma,\end{aligned}$$

We find that $\partial_\rho \bar{h}'^{\mu\rho} = \partial_\rho \bar{h}^{\mu\rho} - \Box^2\xi^\mu$

If we choose… $\Box^2\xi^\mu = \partial_\rho \bar{h}^{\mu\rho}$

Final Linearized Einstein Equation

→ $\Box^2 \bar{h}'_{\mu\nu} = -2\kappa T'_{\mu\nu}$

Towards Relativistic reference frames

Post-Newtonian & Minkowskian approximations

The question : how to represent in practice $|h_{\mu\nu}| << 1$

By series… But which small parameter ?

Weak gravitational field
and slow velocity

Only weak gravitational field $\frac{GM}{c^2 r} << 1$ Post-Minkowskian approx.

Weak gravitational field
and slow velocity $\frac{GM}{c^2 r} << 1 \ , \sqrt{\frac{GM}{c^2 r}} \simeq \frac{v}{c}$ Post-Newtonian approx.

Towards Relativistic reference frames

Post-Newtonian & Minkowskian approximations

Post-Minkowskian approx. $h_{\mu\nu} = \sum G^n h^{(n)}_{\mu\nu}$

Post-Newtonian approx. $h_{\mu\nu} = \sum \frac{1}{c^n} h^{(n)}_{\mu\nu}$

Particular case of the Post-Newtonian approx.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu + \sum_n \frac{1}{c^n} h^{(n)}_{\mu\nu} dx^\mu dx^\nu$$

Motion of a free particule under gravity in Newtonian regime…

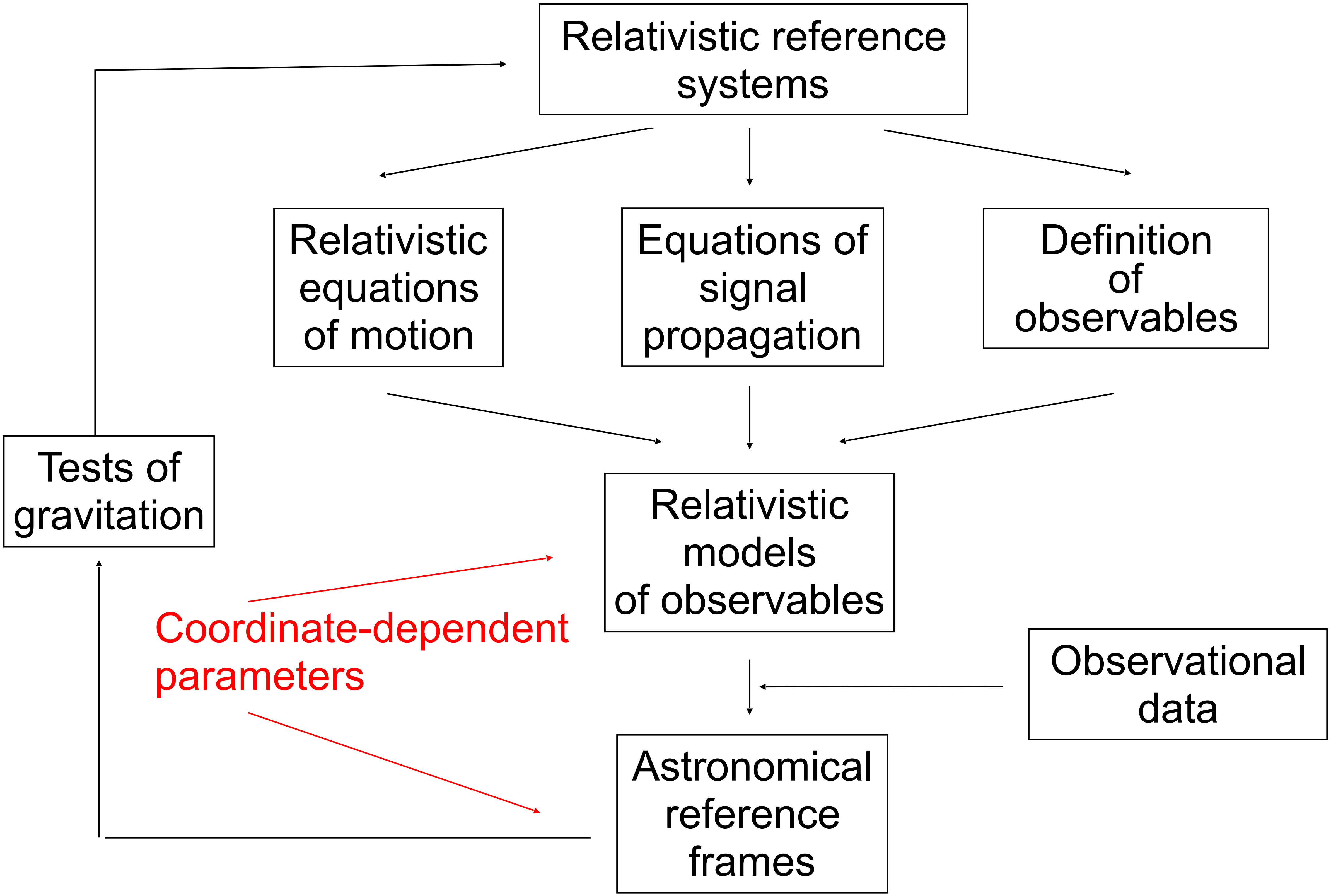
$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0. \quad \text{But} \quad \frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \longrightarrow \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{00} c^2 \left(\frac{dt}{d\tau}\right)^2 = 0$$

Obviously $\frac{d^2 t}{d\tau^2} = 0$ and $\frac{d^2 \vec{x}}{d\tau^2} = -\frac{1}{2} c^2 \left(\frac{dt}{d\tau}\right)^2 \vec{\nabla} h_{00}.$

$$\frac{d^2 \vec{x}}{dt^2} = -\frac{1}{2} c^2 \vec{\nabla} h_{00}$$

$$h_{00} = 2\Phi / c^2.$$

Relativistic Astronomy : some basics



Klioner, 2003

IAU Reference Systems and relativity

- First attempt : IAU 1976
- IAU 2000:
  - Fully relativistic (General Relativity, not PPN)
  - BCRS: time scale TCB
  - GCRS: time scale TCG
  - Time transformation between TCG & TCB
- IAU 2006: redefinition of time scale TDB

THE IAU 2000 RESOLUTIONS FOR ASTROMETRY, CELESTIAL MECHANICS, AND METROLOGY IN THE RELATIVISTIC FRAMEWORK: EXPLANATORY SUPPLEMENT

M. Soffel,[1] S. A. Klioner,[1] G. Petit,[2] P. Wolf,[2] S. M. Kopeikin,[3] P. Bretagnon,[4] V. A. Brumberg,[5] N. Capitaine,[6] T. Damour,[7] T. Fukushima,[8] B. Guinot,[6] T.-Y. Huang,[9] L. Lindegren,[10] C. Ma,[11] K. Nordtvedt,[12] J. C. Ries,[13] P. K. Seidelmann,[14] D. Vokrouhlický,[15] C. M. Will,[16] and C. Xu[17]

Received 2002 August 9; accepted 2003 July 2

[2] Bureau International des Poids et Mesures, Pavillon de Breteuil, F-92312 Sèvres, France.
[3] Department of Physics and Astronomy, 322 Physics Building, University of Missouri–Columbia, Columbia, MO 65211.
[4] Bureau des Longitudes, 77 Avenue Denfert-Rochereau, F-75014 Paris, France.
[5] Institute of Applied Astronomy, Russian Academy of Sciences, Naberezhnaya Kutuzova 10, St. Petersburg 191187, Russia.
[6] Observatoire de Paris, 61 Avenue de l'Observatoire, F-75014 Paris, France.
[7] Institut des Hautes Etudes Scientifiques, 35 Route de Chartres,

oscillation reduces to only a few centimeters (Mashhoon 1985; Soffel, Ruder, & Schneider 1986).

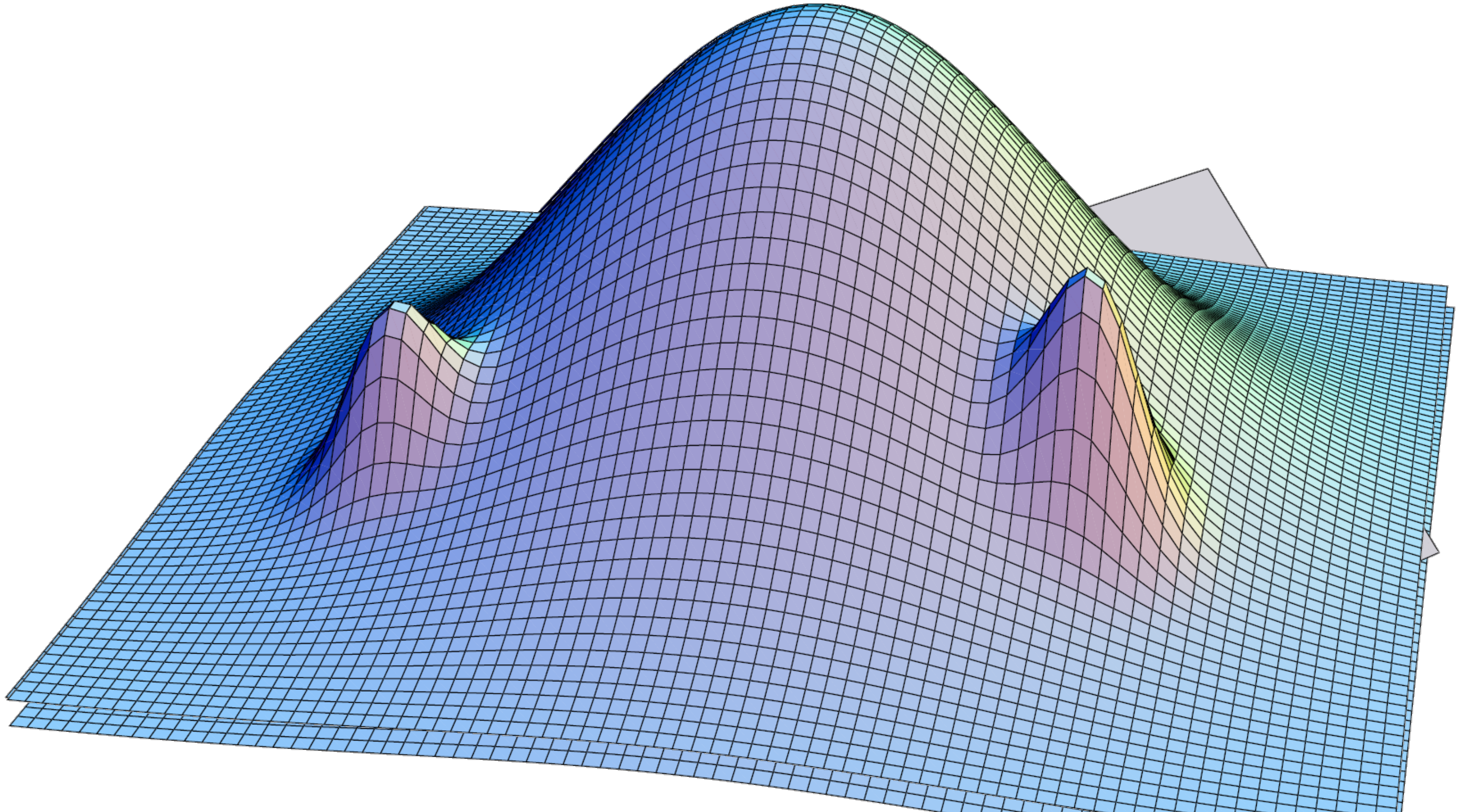
The situation is even more critical in the field of astrometry. It is well known that the gravitational light deflection at the limb of the Sun amounts to 1.″75 and decreases only as $1/r$ with increasing impact parameter r of a light ray to the solar center. Thus, for light rays incident at about 90° from the Sun the angle of light deflection still amounts to 4 mas. To describe the accuracy of astrometric

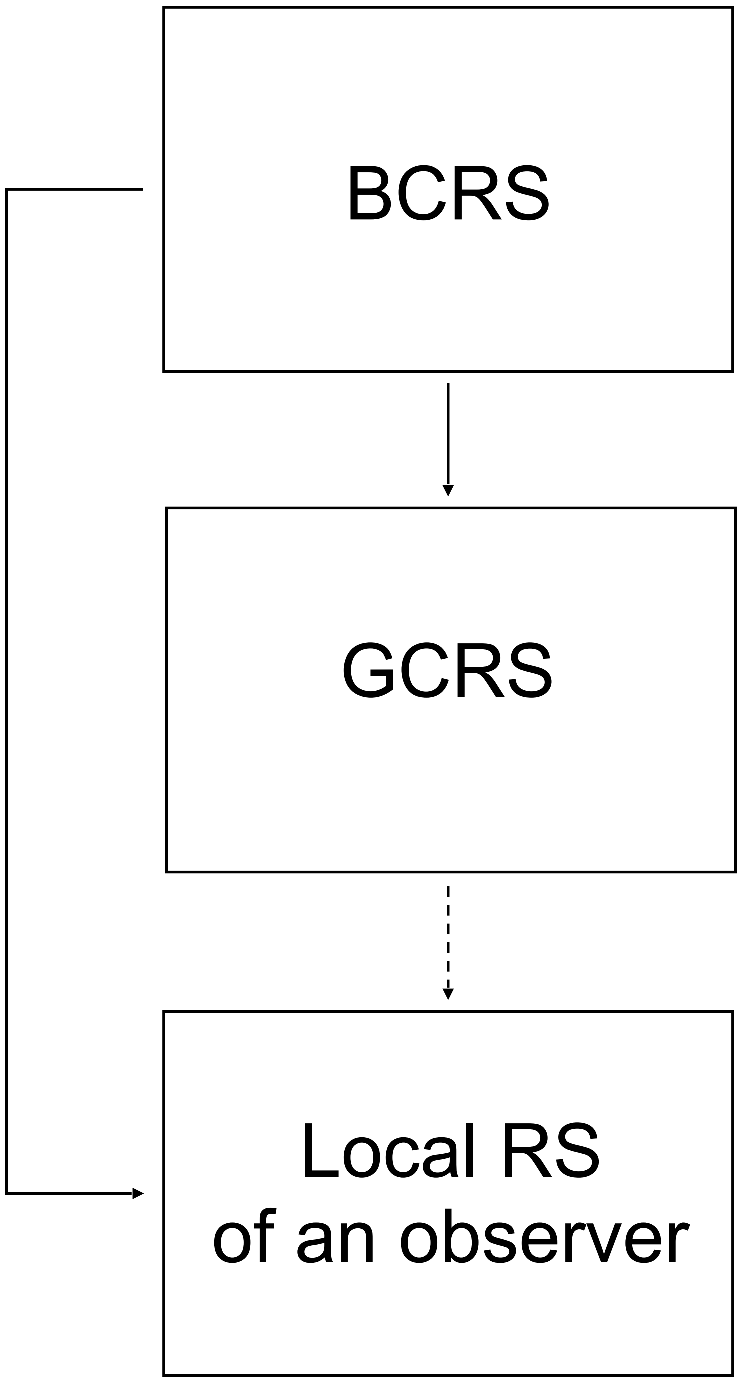
Reference systems theory

- In relativistic astronomy the
  - BCRS (Barycentric Celestial Reference System)
  - GCRS (Geocentric Celestial Reference System)
  - Local reference system of an observer

  play an important role.

- All these reference systems are defined by
  the form of the corresponding metric tensor.





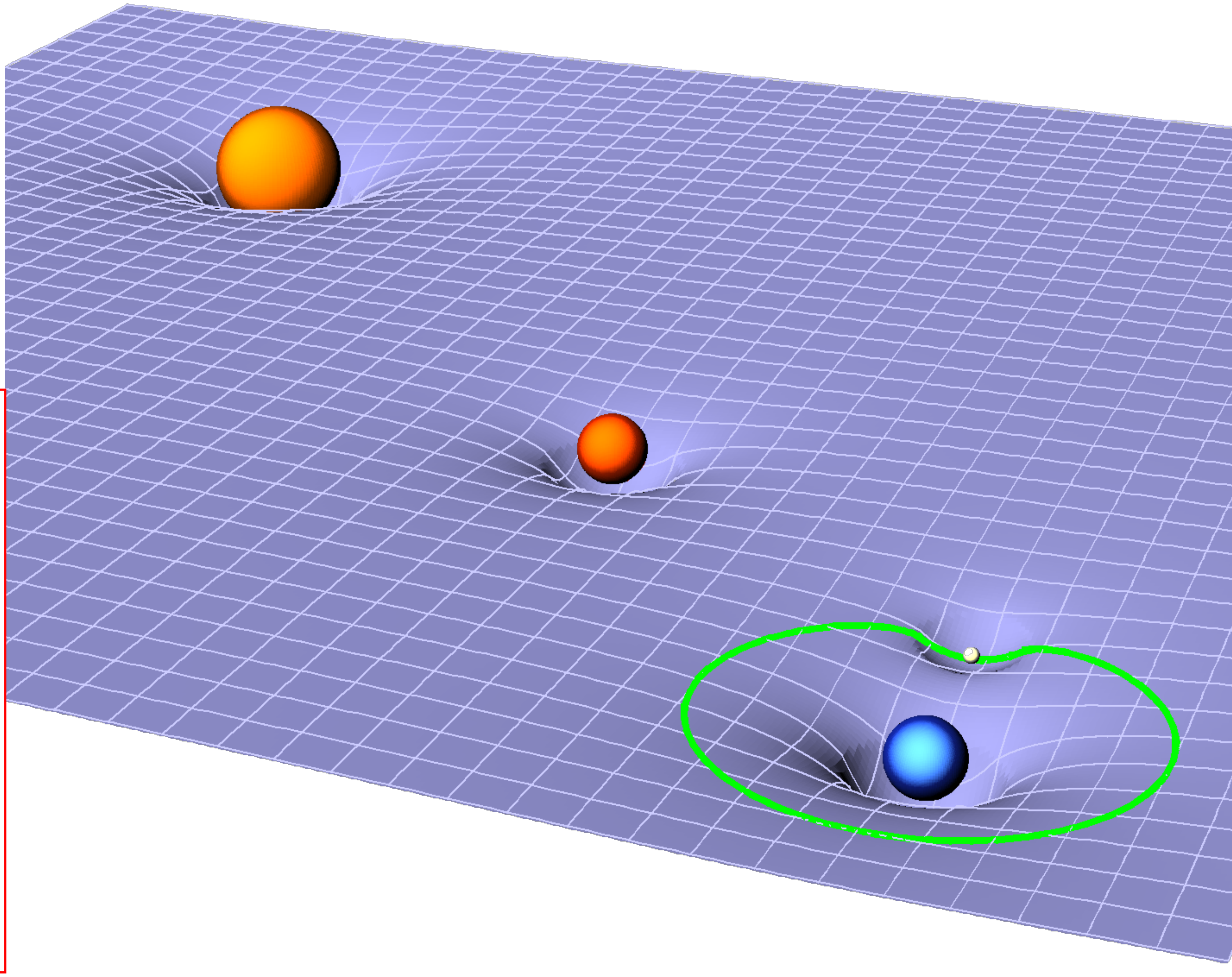
Bini, Crosta & De Felice, 2003
Klioner, 2004

Barycentric Celestial Reference System

The BCRS is a particular reference system in the curved space-time of the Solar system

- One can use any
- but one should fix one :

ICRF by VLBI



$$g_{00} = -1 + \frac{2}{c^2} w(t, \boldsymbol{x}) - \frac{2}{c^4} w^2(t, \boldsymbol{x}) \, ,$$

$$g_{0i} = -\frac{4}{c^3} w^i(t, \boldsymbol{x}) \, ,$$

$$g_{ij} = \delta_{ij} \left(1 + \frac{2}{c^2} w(t, \boldsymbol{x}) \right) .$$

Used to describe motion of celestial body and description of light propagation

Ephemeride — Astrometry